Mathematical statistics

September 20^{th}, 2018

Lecture 8: Point Estimation

Where are we?

Week 1	Probability reviews
Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	**Chapter 7: Point Estimation**
Week 7	Chapter 8: Confidence Intervals
Week 10	Chapter 9: Test of Hypothesis
Week 14	Regression

Overview

7.1 Point estimate
- unbiased estimator
- mean squared error

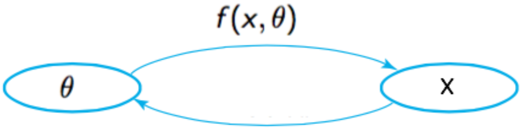
7.2 Methods of point estimation
- method of moments
- method of maximum likelihood.

7.3 Sufficient statistic

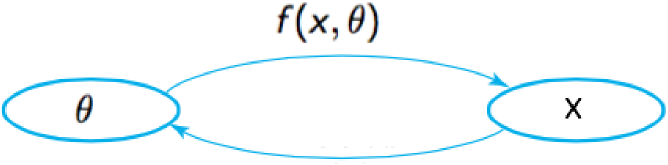
7.4 Information and Efficiency
- Large sample properties of the maximum likelihood estimator

- Bootstrap

- Given a random sample $X_1, \ldots, X_n$ from a distribution with pmf/pdf $f(x, \theta)$ parameterized by a parameter θ
- Goal: Estimate θ



Point estimate



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimate} \\ \theta & \Longrightarrow & X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Example

Problem

Consider a random sample $X_1, \ldots, X_{10}$ from the pdf

$$f(x) = \frac{1+\theta x}{2} \qquad -1 \leq x \leq 1$$

Assume that the obtained data are

$$0.92, \quad -0.1, \quad -0.2, \quad 0.75, \quad 0.65, \quad -0.53$$

$$0.36, \quad -0.68, \quad 0.97, \quad -0.33, \quad 0.79$$

Provide an estimate of θ.

Mean Squared Error

- Measuring error of estimation

$$|\hat{\theta} - \theta| \qquad \text{or} \qquad (\hat{\theta} - \theta)^2$$

- The error of estimation is random

Definition

The mean squared error of an estimator $\hat{\theta}$ is

$$E[(\hat{\theta} - \theta)^2]$$

Problem

Let Y be a random variable and a is a constant. Prove that

$$E[(Y-a)^2] = Var(Y) + (E[Y]-a)^2$$

Hint: Recall that

$$Var[Y] = E[Y^2] - (E[Y])^2$$

Bias-variance decomposition

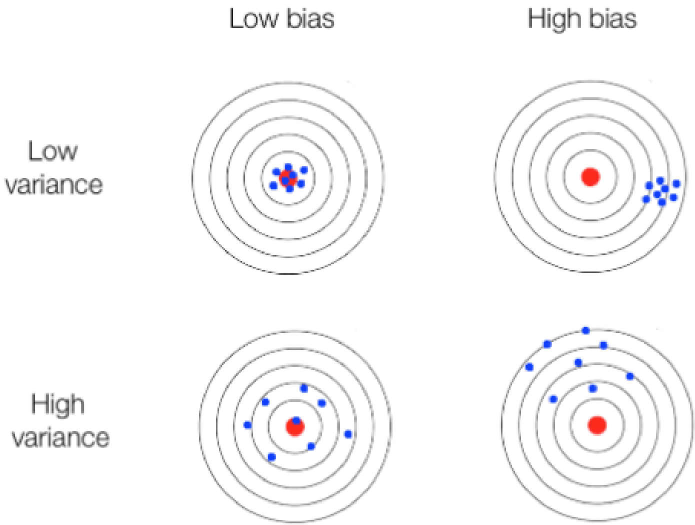
Theorem

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + \left(E(\hat{\theta}) - \theta\right)^2$$

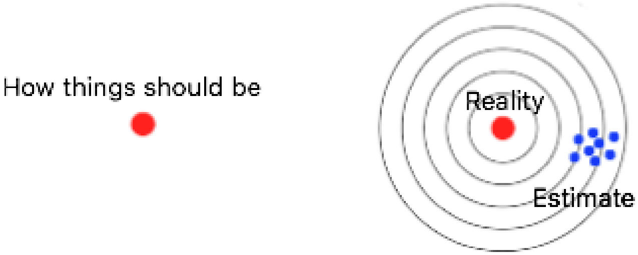
Bias-variance decomposition

Mean squared error $=$ variance of estimator $+$ $(bias)^2$

Bias-variance decomposition



Statistical bias vs. social bias



Unbiased estimators

Definition

A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

for every possible value of θ.

Unbiased estimator

$\Leftrightarrow$ Bias $= 0$

$\Leftrightarrow$ Mean squared error $=$ variance of estimator

Sample proportion

- A test is done with probability of success p. Denote the outcome by let X (success: 1, failure: 0)

$$E[X] = p, \quad Var[X] = p(1-p)$$

- n independent tests are done, let $X_1, X_2, \ldots, X_n$ be the outcomes
- Let
$$\hat{p} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$
- We know that
$$E[\hat{p}] = p$$
thus $\hat{p}$ is an unbiased estimator
- Compute $MSE(\hat{p})$

Sample proportion

- Let

$$\hat{p} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$\begin{aligned} MSE(\hat{p}) &= Var(\hat{p}) + (bias)^2 \\ &= Var\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) \\ &= \frac{1}{n^2}\left(Var(X_1) + Var(X_2) + \ldots + Var(X_n)\right) \\ &= \frac{p(1-p)}{n} \end{aligned}$$

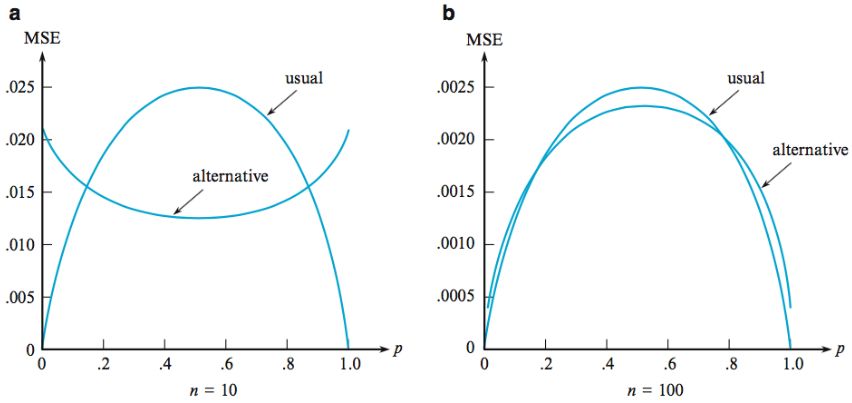
Sample proportion

- A test is done with probability of success p
- n independent tests are done, let $X_1, X_2, \ldots, X_n$ be the outcomes
- Strange idea: How about using

$$\tilde{p} = \frac{X_1 + X_2 + \ldots + X_n + 2}{n + 4}$$

- What is the bias of $\tilde{p}$?
- Compute $MSE(\tilde{p})$

Example 7.1 and 7.4



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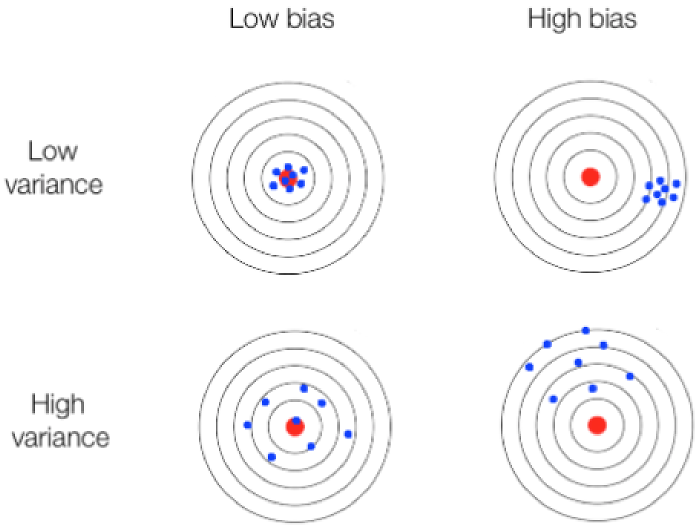
- A test is done with probability of success p
- n independent tests are done, let $X_1, X_2, \ldots, X_n$ be the outcomes
- Crazy idea: How about using

$$\tilde{p} = \frac{X_1 + X_2 + \ldots + X_n + \sqrt{n/4}}{n + \sqrt{n}}$$

This estimator is **always** better than the sample proportion $\hat{p}$

Bias-variance decomposition



Minimum variance unbiased estimator (MVUE)

Definition

Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting $\hat{\theta}$ is called the minimum variance unbiased estimator (MVUE) of θ.

Recall:

- Mean squared error $=$ variance of estimator $+$ $(\textit{bias})^2$
- unbiased estimator $\Rightarrow$ bias $=0$

$\Rightarrow$ MVUE has minimum mean squared error among unbiased estimators