Mathematical statistics

September 27^{th}, 2018

Lecture 10: Point estimation - Method of maximum likelihood.

Where are we?

Week 1	Probability reviews
Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	**Chapter 7: Point Estimation**
Week 7	Chapter 8: Confidence Intervals
Week 10	Chapter 9: Test of Hypothesis
Week 14	Regression

1) Carl Rees Lectures in Mathematical Sciences

Thursday, September 27, 2018, **3:30 PM**

@Kirkbride 204 : Title: **An Approach to Statistical Shape Analysis**

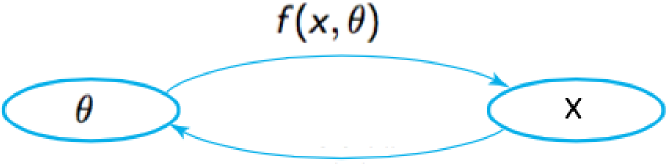
General Talk, **Graduate and Undergraduate Students Welcome**

Speaker: Fadil Santosa, University of Minnesota

Desrcription: In Statistical Shapes Analysis, the goal is to obtain characteristics such as mean, standard deviation, etc., from a set of shapes. While much progress in this area has occurred in the past four decades, many challenges remain. This presentation will review several of the important developments in this field. An approach based on Fourier Analysis is proposed and its capabilities demonstrated.

Overview

Point estimate



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimate} \\ \theta & \Longrightarrow & X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Method of moments

Moments

- Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots,$ the k^{th} population moment, or k^{th} moment of the distribution $f(x)$, is

$$E(X^k)$$

  - First moment: the mean
  - Second moment: $E(X^2)$

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots,$ the k^{th} sample moment is

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n}$$

The law of large numbers provides that when $n \to \infty$

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n} \to E(X^k)$$

Method of moments: ideas

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf
$$f(x; \theta_1, \theta_2, \ldots, \theta_m)$$
- Assume that for $k = 1, \ldots, m$
$$\hat{u}_k = \frac{X_1^k + X_2^k + \ldots + X_n^k}{n} = E(X^k)$$
- Solve the system of equations for $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$

Method of moments: Example

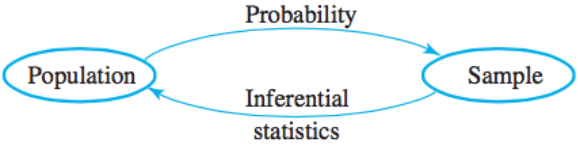
Problem

Let $\beta > 1$ and $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the method of moments to obtain an estimator of β.

Method of maximum likelihood



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

Independent random variables

Definition

Two random variables X and Y are said to be independent if for every pair of x and y values,

$$P(X = x, Y = y) = P_X(x) \cdot P_Y(y) \qquad \text{if the variables are discrete}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad \text{if the variables are continuous}$$

Random sample

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with density function $f_X(x)$.

Then the density of the joint distribution of $(X_1, X_2, ..., X_n)$ is

$$f_{joint}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

Maximum likelihood estimator

- Let $X_1, X_2, ..., X_n$ have joint pmf or pdf

$$f_{joint}(x_1, x_2, \ldots, x_n; \theta)$$

where θ is unknown.

- When $x_1, \ldots, x_n$ are the observed sample values and this expression is regarded as a function of θ, it is called the likelihood function.
- The maximum likelihood estimates θ_{ML} are the value for θ that maximize the likelihood function:

$$f_{joint}(x_1, x_2, \ldots, x_n; \theta_{ML}) \geq f_{joint}(x_1, x_2, \ldots, x_n; \theta) \qquad \forall \theta$$

How to find the MLE?

- Step 1: Write down the likelihood function.
- Step 2: Can you find the maximum of this function?
- Step 3: Try taking the logarithm of this function.
- Step 4: Find the maximum of this new function.

To find the maximum of a function of θ:

- compute the derivative of the function with respect to θ
- set this expression of the derivative to 0
- solve the equation

Example

Problem

Let $X_1, \dots, X_{10}$ be a random sample from the exponential distribution with parameter λ, that is

$$f(x;\lambda) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

The observed data are

$$3.11, 0.64, 2.55, 2.20, 5.44,$$

$$3.42, 1.39, 8.13, 1.82, 1.30$$

Use the method of maximum likelihood to obtain an estimator of λ.

Example

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

A random sample of ten students yields data

$$x_1 = .92,\ x_2 = .79,\ x_3 = .90,\ x_4 = .65,\ x_5 = .86,$$

$$x_6 = .47,\ x_7 = .73,\ x_8 = .97,\ x_9 = .94,\ x_{10} = .77$$

Use the method of maximum likelihood to obtain an estimator of θ.

Example

Problem

Let $X_1, \ldots, X_n$ *be a random sample from the normal distribution* $\mathcal{N}(0, \sigma^2)$*, that is*

$$f(x, \theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Use the method of maximum likelihood to obtain an estimator of σ*.*

Example

Problem

Let $\beta > 1$ and $X_1, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the method of maximum likelihood to obtain an estimator of β.