Mathematical statistics

October 11^{th}, 2018

Lecture 14: Confidence intervals

Countdown to mid-term exam: 14 days

Week 1	Chapter 1: Descriptive statistics
Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	Chapter 7: Point Estimation
Week 7	**Chapter 8: Confidence Intervals**
Week 10	Chapter 9: Test of Hypothesis
Week 13	Two-sample inference, ANOVA, regression

Overview

8.1 Basic properties of confidence intervals (CIs)
- Interpreting CIs
- General principles to derive CI

8.2 Large-sample confidence intervals for a population mean
- Using the Central Limit Theorem to derive CIs

8.3 Intervals based on normal distribution
- Using Student's t-distribution

8.4 CIs for standard deviation

Confidence Intervals

Framework

- Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution $f(x, \theta)$
- In Chapter 7, we learnt methods to construct an estimate $\hat{\theta}$ of θ
- Goal: we want to indicate the degree of uncertainty associated with this random prediction
- One way to do so is to construct a *confidence interval* $[\hat{\theta} - a, \hat{\theta} + b]$ such that

$$P[\theta \in [\hat{\theta} - a, \hat{\theta} + b]] = 95\%$$

Principles for deriving CIs

If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution $f(x, \theta)$, then

- Find a random variable $Y = h(X_1, X_2, \ldots, X_n; \theta)$ such that the probability distribution of Y does not depend on θ or on any other unknown parameters.
- Find constants a, b such that

$$P\left[a < h(X_1, X_2, \ldots, X_n; \theta) < b\right] = 0.95$$

- Manipulate these inequalities to isolate θ

$$P\left[\ell(X_1, X_2, \ldots, X_n) < \theta < u(X_1, X_2, \ldots, X_n)\right] = 0.95$$

95% confidence interval of the mean

- Assumptions:
  - Normal distribution
  - σ is known
- 95% confidence interval
  If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

Assumptions

- Section 8.1
  - Normal distribution
  - σ is known
- Section 8.2
  - ~~Normal distribution~~
  - σ is known

$\rightarrow$ Using Central Limit Theorem $\rightarrow$ needs $n > 30$

- Section 8.3
  - Normal distribution
  - ~~σ is known~~

$\rightarrow$ Introducing t-distribution

$100(1-\alpha)\%$ confidence interval

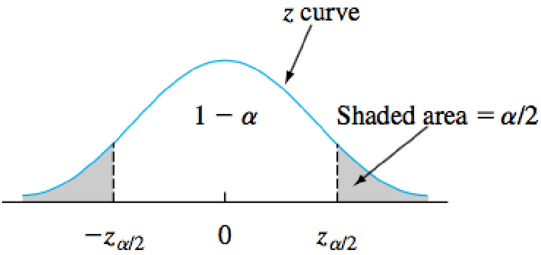


Figure 8.4 $P(-z_{\alpha/2} \le Z \le z_{\alpha/2}) = 1-\alpha$

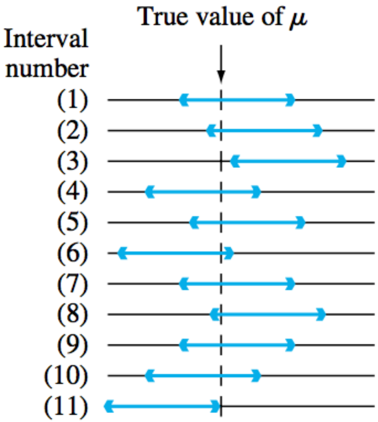
$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Interpreting confidence intervals

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT okay.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is okay.
- Saying "if we repeat the experiment many times, the interval contains μ about 95% of the time" is perfect.

8.2: Large-sample CIs of the population mean

Principles

- Central Limit Theorem
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$
is approximately normal when $n > 30$
- Moreover, when n is sufficiently large $s \approx \sigma$
- Conclusion:
$$\frac{\bar{X} - \mu}{s/\sqrt{n}}$$
is approximately normal when n is sufficiently large

If $n > 40$, we can ignore the normal assumption and replace σ by s

95% confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

One-sided CIs (Confidence bounds)

One-sided CIs

A **large-sample upper confidence bound for** μ is

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

and a **large-sample lower confidence bound for** μ is

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}$$

CIs vs. one-sided CIs

CIs:

- $100(1-\alpha)\%$ confidence

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

One-sided CIs:

- $100(1-\alpha)\%$ confidence

$$\left(-\infty, \bar{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(-\infty, \bar{x} + 1.64\frac{\sigma}{\sqrt{n}}\right)$$

Confidence level

Problem

Determine the confidence level for each of the following large-sample confidence intervals/bounds:

(a) $\bar{x} + 0.84s/\sqrt{n}$

(b) $\left(\bar{x} - 0.84s/\sqrt{n}, \bar{x} + 0.84s/\sqrt{n}\right)$

(c) $\bar{x} - 2.05s/\sqrt{n}$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

8.3: Intervals based on normal distributions

Assumptions

- the population of interest is normal (i.e., $X_1, \dots, X_n$ constitutes a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$).
- σ is unknown

$\rightarrow$ we want to consider cases when n is small.

Principles

- When $n < 40$, S is no longer close to σ. Thus

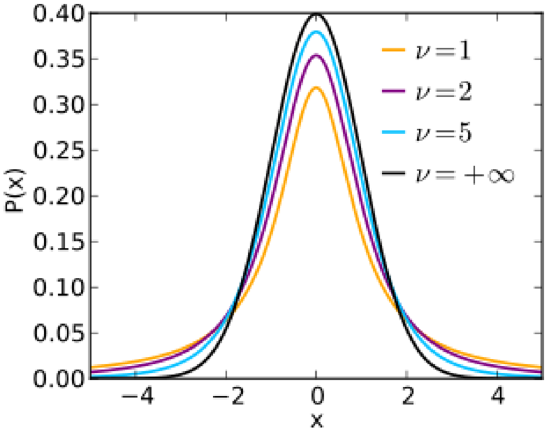
$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

  does not follow the standard normal distribution.
- {Section 6} But since we know the distribution of X, technically we can compute the distribution of T
- Moreover, the distribution of T does not depend on μ and σ {More reading: Section 6.4}

t distributions with degree of freedom ν

Probability density function

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$



t distributions

PROPERTIES OF T DISTRIBUTIONS

1. Each t_ν curve is bell-shaped and centered at 0.
2. Each t_ν curve is more spread out than the standard normal (z) curve.
3. As ν increases, the spread of the t_ν curve decreases.
4. As $\nu \to \infty$, the sequence of t_ν curves approaches the standard normal curve (so the z curve is often called the t curve with df $= \infty$).

t distributions

When $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ, the rv

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has the t distribution with $n - 1$ degree of freedom (df).

t distributions

Let $t_{\alpha,\nu}$ = the number on the measurement axis for which the area under the t curve with ν df to the right of $t_{\alpha,\nu}$ is α; $t_{\alpha,\nu}$ is called a **t critical value.**

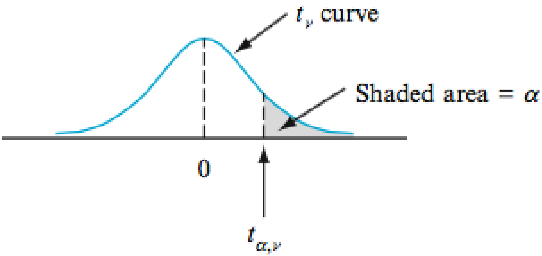


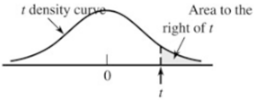
Figure 8.7 A pictorial definition of $t_{\alpha,\nu}$

How to do computation with t distributions

- Instead of looking up the normal Z-table A3, look up the two t-tables A5 and A7.
- Idea
$$P[T \geq t_{\alpha,\nu}] = \alpha$$
- $\{$From t, find $\alpha\}$ $\rightarrow$ using table A7
- $\{$From α, find $t\}$ $\rightarrow$ using table A5

$t \to \alpha$

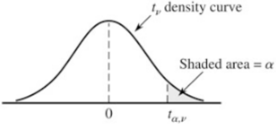
Table A.7 *t* Curve Tail Areas



t \ ν	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0.0	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500
0.1	.468	.465	.463	.463	.462.	.462	.462	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461
0.2	.437	.430	.427	.426	.425	.424	.424	.423	.423	.423	.423	.422	.422	.422	.422	.422	.422	.422
0.3	.407	.396	.392	.390	.388	.387	.386	.386	.386	.385	.385	.385	.384	.384	.384	.384	.384	.384
0.4	.379	.364	.358	.355	.353	.352	.351	.350	.349	.349	.348	.348	.348	.347	.347	.347	.347	.347
0.5	.352	.333	.326	.322	.319	.317	.316	.315	.315	.314	.313	.313	.313	.312	.312	.312	.312	.312
0.6	.328	.305	.295	.290	.287	.285	.284	.283	.282	.281	.280	.280	.279	.279	.279	.278	.278	.278
0.7	.306	.278	.267	.261	.258	.255	.253	.252	.251	.250	.249	.249	.248	.247	.247	.247	.247	.246
0.8	.285	.254	.241	.234	.230	.227	.225	.223	.222	.221	.220	.220	.219	.218	.218	.218	.217	.217
0.9	.267	.232	.217	.210	.205	.201	.199	.197	.196	.195	.194	.193	.192	.191	.191	.191	.190	.190
1.0	.250	.211	.196	.187	.182	.178	.175	.173	.172	.170	.169	.169	.168	.167	.167	.166	.166	.165
1.1	.235	.193	.176	.167	.162	.157	.154	.152	.150	.149	.147	.146	.146	.144	.144	.144	.143	.143
1.2	.221	.177	.158	.148	.142	.138	.135	.132	.130	.129	.128	.127	.126	.124	.124	.124	.123	.123
1.3	.209	.162	.142	.132	.125	.121	.117	.115	.113	.111	.110	.109	.108	.107	.107	.106	.105	.105
1.4	.197	.148	.128	.117	.110	.106	.102	.100	.098	.096	.095	.093	.092	.091	.091	.090	.090	.089
1.5	.187	.136	.115	.104	.097	.092	.089	.086	.084	.082	.081	.080	.079	.077	.077	.077	.076	.075
1.6	.178	.125	.104	.092	.085	.080	.077	.074	.072	.070	.069	.068	.067	.065	.065	.065	.064	.064
1.7	.169	.116	.094	.082	.075	.070	.065	.064	.062	.060	.059	.057	.056	.055	.055	.054	.054	.053
1.8	.161	.107	.085	.073	.066	.061	.057	.055	.053	.051	.050	.049	.048	.046	.046	.045	.045	.044
1.9	.154	.099	.077	.065	.058	.053	.050	.047	.045	.043	.042	.041	.040	.038	.038	.038	.037	.037

$\alpha \to t$

Table A.5 Critical Values for t Distributions



	α						
ν	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965

Let $\overline{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal population with mean μ. Then a **100(1 − α)% confidence interval for μ**, the **one-sample t CI**, is

$$\left(\overline{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \overline{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}\right) \tag{8.15}$$

or, more compactly, $\overline{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$.

An **upper confidence bound for μ** is

$$\overline{x} + t_{\alpha,n-1} \cdot \frac{s}{\sqrt{n}}$$

and replacing + by − in this latter expression gives a **lower confidence bound for μ**; both have confidence level 100(1 − α)%.

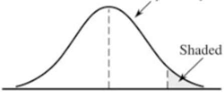
Practice problem

31. Determine the t critical value for a two-sided confidence interval in each of the following situations:

a. Confidence level = 95%, df = 10

b. Confidence level = 95%, df = 15

Table A.5 Critical Values for *t* Distributions



	α						
ν	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965

Problem

Here are the lengths (in minutes) of the 63 nine-inning games from the first week of the 2001 major league baseball season:

194	160	176	203	187	163	162	183	152	177
177	151	173	188	179	194	149	165	186	187
187	177	187	186	187	173	136	150	173	173
136	153	152	149	152	180	186	166	174	176
198	193	218	173	144	148	174	163	184	155
151	172	216	149	207	212	216	166	190	165
176	158	198							

Assume that this is a random sample of nine-inning games (the mean differs by 12 s from the mean for the whole season).

a. Give a 95% confidence interval for the population mean.

b. Give a 95% prediction interval for the length of the next nine-inning game. On the first day of the next week, Boston beat Tampa Bay 3–0 in a nine-inning game of 152 min. Is this within the prediction interval?

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997