MATH 450 **Name (Print):** ____________________

Fall 2017

Review problems

12/7/17

Time Limit: 90 Minutes

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books/notes on this exam. You may use calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	40	
2	40	
3	40	
4	40	
5	40	
Total:	200	

1. (40 points) A gas station sells three grades of gasoline: regular unleaded, extra unleaded, and super unleaded. These are priced at 2.2, 2.3, and 2.5 per gallon, respectively.

   Let X_1, X_2, and X_3 denote the amounts of these grades purchased (gallons) on a particular day. Suppose the X_i's are independent and are *normally distributed* with $\mu_1 = 1000$, $\mu_2 = 500$, $\mu_3 = 300$, $\sigma_1 = 100$, $\sigma_2 = 80$, $\sigma_3 = 50$. What is the probability that the revenue

   $$Y = 2.2X_1 + 2.3X_2 + 2.5X_3.$$

   exceeds 4500?

   We first compute the expected value and variance of Y.

   $$\mu_Y = 2.2E[X_1] + 2.3E[X_2] + 2.5E[X_3] = 4100,$$
   $$V[Y] = 2.2^2 V[X_1] + 2.3^3 V[X_2] + 2.5^2 V[X_3] = 97881$$

   and $\sigma_Y = 312.86$.

   Thus,

   $$\begin{aligned} P[Y > 4500] &= P\left[\frac{Y - \mu_Y}{\sigma_Y} > \frac{4500 - 4100}{312.86}\right] \\ &= P\left[Z > 1.2785\right] \\ &= 1 - P\left[Z \leq 1.2785\right] \\ &= 1 - 0.8997 = 0.1003 \end{aligned}$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with population mean μ and population standard deviation σ^2. Let $\bar{X}$ and S denote the sample mean and sample standard deviation.

   (a) (20 points) Show that $(\bar{X})^2$ is not an unbiased estimator of μ^2.
   Hint: For any random variable Y, we have $E(Y^2) = V(Y) + [E(Y)]^2$. Apply this with $Y = \bar{X}$.

   Recalling that

   $$E[\bar{X}] = \mu, \quad V[\bar{X}] = \frac{\sigma^2}{n},$$

   we have

   $$E[(\bar{X})^2] = V[\bar{X}] + (E[\bar{X}])^2 = \frac{\sigma^2}{n} + \mu^2 \neq \mu^2$$

   Thus, $(\bar{X})^2$ is not an unbiased estimator of μ^2.

   (b) (20 points) Recalling that S^2 is an unbiased estimator of σ^2, for what value of k is the estimator $(\bar{X})^2 - kS^2$ is an unbiased estimator of μ^2?
   Hint: Compute $E[(\bar{X})^2 - kS^2]$ and find k to make it equal to μ^2.

   Since S^2 is an unbiased estimator of σ^2, we have $E[S^2] = \sigma^2$. Thus,

   $$E[(\bar{X})^2 - kS^2] = E[(\bar{X})^2] - kE[S^2] = \frac{\sigma^2}{n} + \mu^2 - k\sigma^2.$$

   The estimator $(\bar{X})^2 - kS^2$ is an unbiased estimator of μ^2 if and only if $E[(\bar{X})^2 - kS^2] = \mu^2$, which is correct if we choose $k = 1/n$.

3. Suppose that against a certain opponent, the number of points a basketball team scores is normally distributed with unknown mean μ and unknown variance σ^2. Suppose that over the course of the last 10 games, the team scored the following points:

$$59, \quad 62, \quad 59, \quad 74, \quad 70, \quad 61, \quad 62, \quad 66, \quad 62, \quad 75$$

(a) (20 points) Compute a 95% confidence interval for μ.

We have $\bar{x} = 64, s = 6.07$ and $t_{0.025,9} = 2.262$

$$\left(\bar{x} - t_{0.025,9}\frac{s}{\sqrt{n}}, \bar{x} + t_{0.025,9}\frac{s}{\sqrt{n}}\right)$$

(b) (20 points) Now suppose that you learn that $\sigma^2 = 25$. Compute a 95% confidence interval for μ. How does this compare to the interval in (a)?

We have $\bar{x} = 64, \sigma = 5$ and $z_{0.025} = 1.96$

$$\left(\bar{x} - z_{0.025}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}}\right)$$

4. A company that makes cola drinks states that the mean caffeine content per one 12-ounce bottle of cola is 40 milligrams. You work as a quality control manager and are asked to test this claim. During your tests, you find that a random sample of 30 bottles of cola (12-ounce) has a mean caffeine content of 39.2 milligrams. From a previous study, you know that the standard deviation of the population is $\sigma = 7.5$ milligrams. We assume that the caffeine content is normally distributed.

   (a) (20 points) At α= 1% level of significant, can you reject the company's claim? What is the P-value associated with the test?

   Denote the mean caffein content by μ. We are testing

   $$H_0 : \mu = 40$$
   $$H_a : \mu \neq 40$$

   Since σ is given, we can use a z-statustuc

   $$z = \frac{\bar{x} - \Delta_0}{\sigma/\sqrt{n}} = \frac{39.2 - 40}{7.5/\sqrt{30}} = -0.584$$

   The corresponding p-value of the test is 0.559. Conclusion: we have no evidence to reject the null hypothesis. We dont have enough evidence to reject the claim that the mean caffeine level of all 12-ounce bottle of cola is 40 milligrams.

   (b) (20 points) Assume that another bottle of cola is randomly sampled from the same population. Construct a 95% prediction interval for the caffeine content of that bottle.

   The 95% prediction interval for X_{31} is

   $$\left(\bar{x} - z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}}, \bar{x} + z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}}\right) = (24.26, 54.16)$$

5. A physician claims that an experimental medication *increases* an individuals heart rate. Twelve test subjects are randomly selected, and the heart rate of each is measured. The subjects are then injected with the medication and, after 1 hour, the heart rate of each is measured again. The results are show below.

Before	72	81	76	74	75	80	68	75	78	76	74	77
After	73	80	79	76	76	80	74	77	75	74	76	78

We assume that both samples were selected from normal distributions.

(a) (20 points) Carry out a test of hypotheses with significance level $\alpha = 0.05$ to decide whether we should reject the physician's claim. Provide the P-value of the test.

Note: This is paired data, since its a before-after experiment. Since we want to prove that the medication increases the heart rate, the difference of Before After should be negative.

Denote by μ_1 and μ_2 the mean heart rate before and after the treatment, respectively. Let $\mu_d = \mu_1 - \mu_2$, we want to test

$$H_0 : \mu_d = 0$$
$$H_a : \mu_d < 0$$

where

$$D = [-1, 1, -3, -2, -1, 0, -6, -2, 3, 2, -2, -1]$$

We use a t-test with degree of freedom $\nu = n - 1 = 11$

$$t = \frac{\bar{x} - \Delta_0}{s/\sqrt{n}} = \frac{-1}{2.37/\sqrt{12}} = -1.46$$

The corresponding p-value of the test is 0.081 (or 0.095, depending on how you round the t-value to look up on the table). Since p-value $> 5\%$, we fail to reject the null hypothesis. We don't have enough evident to conclude that the experimental medication *increases* individuals heart rates.

(b) (20 points) Another physician claims that the experimental medication above does *change* the heart rate. Do we have enough evidence to reject his claim, using the same significance level $\alpha = 0.05$?

We want to test

$$H_0 : \mu_d = 0$$
$$H_a : \mu_d \neq 0$$

The t-statistic value is the same, but the p-value is calculated differently. In this case, we have p-value$= 2 \times 0.081 = 0.162$. We also don't have enough evident to conclude that the experimental medication *changes* individuals heart rates.