Answer key

(1a) 0.0605
(Law of total probability)

(1b) 0.4628
(Bayes' formula)

(2a) \$700
The profit can computed by

$$h(X) = revenue - cost = 1000X + 200(3 - X) - 1500 = 800X - 900$$

You just need to compute $E[h(X)]$

(2b) We have $X \sim B(n = 25, p = 0.05)$, thus

$$\begin{aligned} P[X \leq 2] &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \binom{25}{0}(1-p)^{25} + \binom{25}{1} p \times (1-p)^{24} + \binom{25}{2}(1-p)^{23}p^2 \\ &= 0.27739 + 0.36499 + 0.23052 \\ &= 0.8729 \end{aligned}$$

(3a) $c = 2$

(3b) $P[1/4 \leq X \leq 3/4] = 1/2$; $Var(X) = 1/18$.

(3c) For $0 < a < 1$, we have

$$P[X \leq a] = \int_0^a 2(1-x) = 2(a - \frac{a^2}{2})$$

The equation

$$2(a - \frac{a^2}{2}) = \frac{15}{16}$$

is equivalent to

$$\left(a - \frac{3}{4}\right)\left(a - \frac{5}{4}\right) = 0$$

Note that $5/4 > 1$, thus $P[X \leq 5/4] = 1$ and $5/4$ cannot be a solution. We deduce that $a = 3/4$.

(4a) The total cost of the contents of a can is

$$h(X, Y) = 2X + 3Y + (1 - X - Y) = 1 + X + 2Y$$

Thus, the expected cost is

$$E[h(X, Y) = \int_0^1 \int_0^{1-x} (1 + x + 2y) \cdot (24xy) dy\ dx = 2.20$$

(4b, c) Answers:

$$f_X(x) = \begin{cases} 12x(1-x)^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$E(X) = E[Y] = 2/5, \quad E[XY] = 2/15$$

$$Cov(X, Y) = -\frac{2}{75}$$

(5a) 1/9

(5b) Let $C = A \cup B$ and $D = A \cup B^c$, then $C \cap D = A$ and $C \cup D = S$.
Use $P(C \cap D) = P(C) + P(D) - (C \cup D)$, we deduce that $P(A) = 0.6$.