Probability Theory and Simulation Methods

Feb 7th, 2018

Lecture 2: Non-transitive dice

Topics

Week 1	**Chapter 1: Axioms of probability**
Week 2	Chapter 3: Conditional probability and independence
Week 3	Chapters 4,5,6,7: Random variables
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Overview

1. Sample space and events
2. Axioms of probability
3. Other topics
   - Continuity of probability functions
   - Probabilities 0 and 1

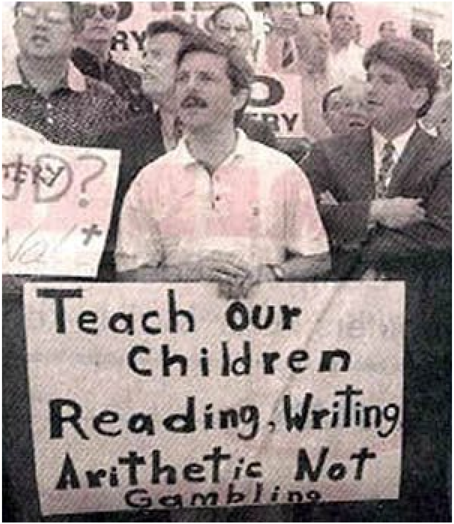
Probability and gambling

Probability and gambling

Modern probability started when Pascal and Fermat discussed gambling

Monte Carlo method

Against teaching gambling



How to gamble (according to mathematicians)

1. Consider all possible outcomes
2. Assess how likely each outcome will happen
3. Choose the course of action that benefits you the most
4. Profit!

Sample space and events

1. An experiment: is any action, process, or phenomenon whose outcome is subject to uncertainty
2. An outcome: is a result of an experiment
   Each run of the experiment results in one outcome
3. A sample space: is the set of all possible outcomes of an experiment
4. An event: is a subset of the sample space.
   An event occurs when one of the outcomes that belong to it occurs

Sample space and events: example

1. Experiment: Toss a coin
2. Outcome: either head (H) or tail (T)
3. Sample space: $\{H, T\}$
4. Events: $\{H, T\}$, $\{H\}$, $\{T\}$, $\emptyset$

Sample space and events: example

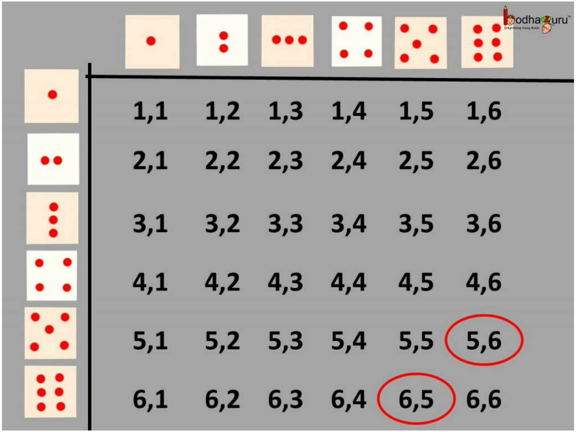
1. Experiment: Toss a coin 2 times
2. Sample space: $\{HH, HT, TH, TT\}$
3. Events: There are 16 different events. Examples:
    - E_1 = the result of the two tosses are different = $\{HT, TH\}$
    - E_2 = the result of the second toss is head= $\{HH, TH\}$

Sample space and events: example

1. Experiment: Toss a regular dice
2. Sample space: $\{1, 2, 3, 4, 5, 6\}$
3. Some events
   - E_1 = the result is an even number $= \{2, 4, 6\}$
   - E_2 = the result is greater than 2$= \{3, 4, 5, 6\}$
   - $E_3 = \{1, 3, 5, 6\}$

Sample space and events: example

1. Experiment: Toss two regular dice
2. Event E_1 = the summation of the two dice is 11



Define probability

1. Experiment: Toss a FAIR coin
2. Outcome: either head (H) or tail (T), each with probability 0.5
3. Sample space: $\{H, T\}$
4. Events: $\{H, T\}$, $\{H\}$, $\{T\}$, $\emptyset$
5. Define:

$$P[\{H\}] = P[\{T\}] = 1/2, P(\emptyset) = 0, P(\{H, T\}) = 1$$

Define probability

1. Experiment: Toss a FAIR coin TWO times
2. Outcome:

$$P(\{HH\}) = P(\{HT\}) = P(\{TH\}) = P(\{TT\}) = 1/4$$

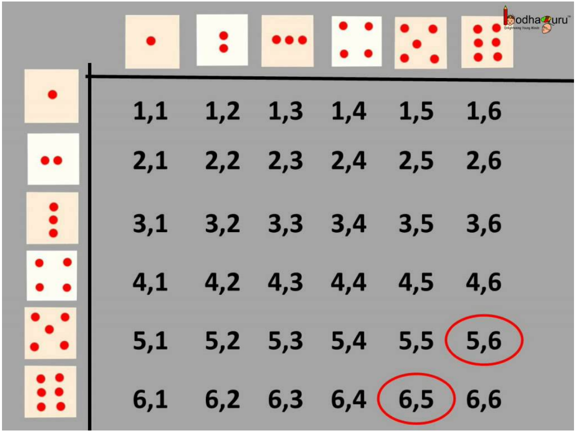
3. Sample space: $\{HH, HT, TH, TT\}$
   - E_1 = the results of the two coins are different = $\{HT, TH\}$
   - E_2 = the result of the second coin is head= $\{HH, TH\}$
4. Thus

$$P[E_1] = P(\{HT\}) + P(\{TH\}) = 1/2$$

Define probability

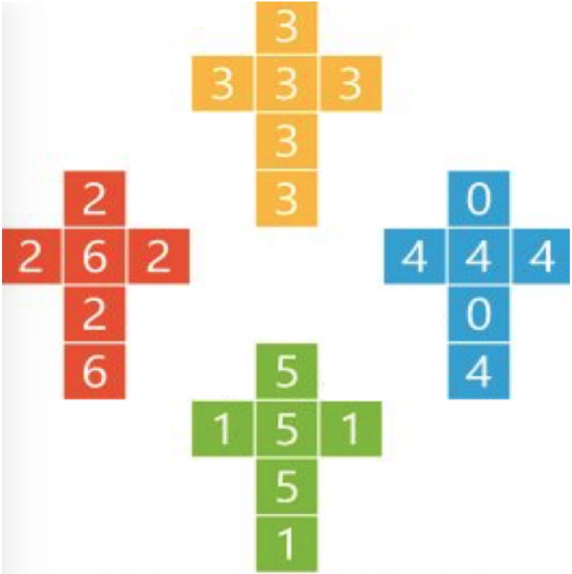
1. Experiment: Toss two regular dice
2. E_1 = the summation of the two dice is 11

$$P[E_1] = 1/18$$

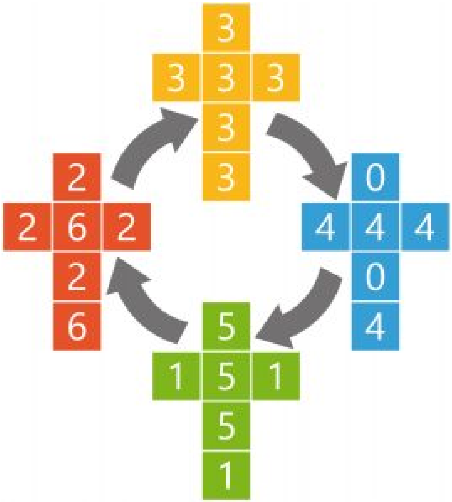


1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

A set of weird dice



Non-transitive dice



What conditions should we impose to define probability?

- P[the sample space] $= 1$
- $0 \leq P[E] \leq 1$ for all events E
- $P[\emptyset] = 0$
- If E_1 and E_2 are disjoint then $P[E_1 \cup E_2] = P[E_1] + P[E_2]$
- If E_1, E_2 and E_3 are mutually disjoint then

$$P[E_1 \cup E_2 \cup E_3] = P[E_1] + P[E_2] + P[E_3]$$

Axioms of probability

Definition (Probability Axioms) *Let S be the sample space of a random phenomenon. Suppose that to each event A of S, a number denoted by $P(A)$ is associated with A. If P satisfies the following axioms, then it is called a* ***probability*** *and the number $P(A)$ is said to be the* ***probability of A.***

Axiom 1 $P(A) \geq 0.$

Axiom 2 $P(S) = 1.$

Axiom 3 *If $\{A_1, A_2, A_3, \ldots\}$ is a sequence of mutually exclusive events (i.e., the joint occurrence of every pair of them is impossible: $A_i A_j = \emptyset$ when $i \neq j$), then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$