Probability Theory and Simulation Methods

Feb 12nd, 2018

Lecture 4: The lesser-known Annus Mirabilis paper

Topics

Week 1	**Chapter 1: Axioms of probability**
Week 2	Chapter 3: Conditional probability and independence
Week 3	Chapters 4,5,6,7: Random variables
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

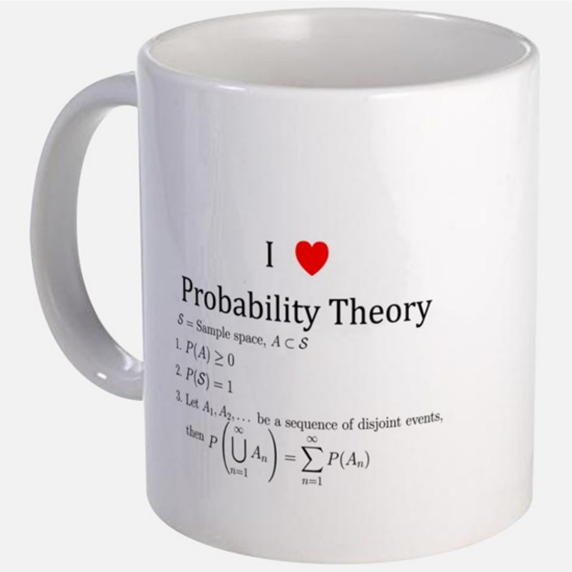
Overview

1. Sample space and events
2. Axioms of probability
3. Other topics

Sample space and events

1. An experiment: is any action, process, or phenomenon whose outcome is subject to uncertainty
2. An outcome: is a result of an experiment
   Each run of the experiment results in one outcome
3. A sample space: is the set of all possible outcomes of an experiment
4. An event: is a subset of the sample space.
   An event occurs when one of the outcomes that belong to it occurs

Axioms of probability



Some basic properties

Theorem

The probability of the empty set is 0.

Theorem

Let A_1, A_2 *be two disjoint events (that is* $A_1 A_2 = \emptyset$*). Then*

$$P[A_1 \cup A_2] = P[A_1] + P[A_2].$$

Other corollaries

Theorem

Let A^c be the complement event of A. Then

$$P[A^c] = 1 - P[A].$$

Theorem

Let $A \subset B$, then $P[A] \leq P[B]$.

Problem 1

Problem

For any events A, B

$$P[A \cup B] = P[A] + P[B] - P[AB]$$

Problem 2

Problem

For any events A, B

$$P[A] = P[AB] + P[AB^c]$$

What we have learnt so far

1. Sample space and events
   [with examples for which the sample space is finite]
2. Axioms of probability
   [the same space is an abstract set]

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

To infinity and beyond

Random selection of points from an interval

Assume that

- I'm going to select a point from the interval $[-1, 1]$, randomly
- Every point is equally likely to be chosen

$\rightarrow$ the sample space is $[-1, 1]$, each point $x \in [-1, 1]$ is an outcome, an event is a subset of $[-1, 1]$

Question: What is $P([0, 1])$ the probability that the chosen point belong to the interval $[0, 1]$?

Random selection of points from an interval

Assume that

- I'm going to select a point from the interval $[-1, 1]$, randomly
- Every point is equally likely to be chosen

Question:

(a) What is $P([-2/3, -1/3] \cup [1/4, 3/4])$

(b) Let m be a natural number, what is $P([-1/m, 1/m])$?

(c) What is $P(\{0\})$?

Probabilities 0 and 1

Assume that

- I'm going to select a point from the interval $[-1, 1]$, randomly
- Every point is equally likely to be chosen

then $P(\{a\}) = 0$ for all $a \in [-1, 1]$.

Remark: for an event E, $P(E) = 0$ does not mean that it cannot occur

Limit of decreasing events

- A sequence $\{E_n, n \geq 1\}$ of events of a sample space is called decreasing if

$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots$$

- Definition: by $\lim_{n\to\infty} E_n$ we mean the event that at least one E_i occurs, that is

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$$

- Example: In the previous example, let $E_n = [-1/n, 1/n]$, then

$$\lim E_n = \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} [-1/n, 1/n] = \{0\}$$

Limit of increasing events

Theorem 1.8 (Continuity of Probability Function) *For any increasing or decreasing sequence of events,* $\{E_n, n \geq 1\}$,

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n).$$

In the previous example, let $E_n = [-1/n, 1/n]$, then

- $\lim E_n = \{0\}$
- $P(E_n) = 1/n$, $P(\{0\}) = 0$

To infinity and beyond

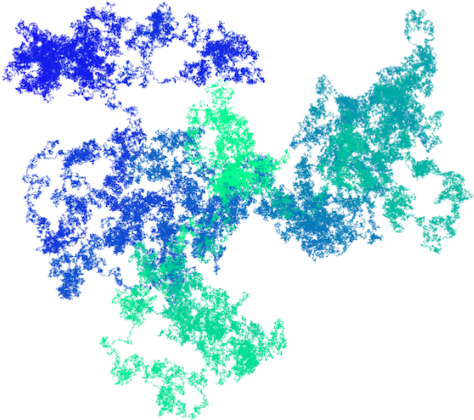
- 26 years old
- just finished PhD
- worked as an examiner at the Patent Office in Bern
- ~~supported himself by playing poker~~ "God does not play dice with the universe"

1905: The Wonderful Year

This year, Einstein published 4 papers, all on Annalen der Physik

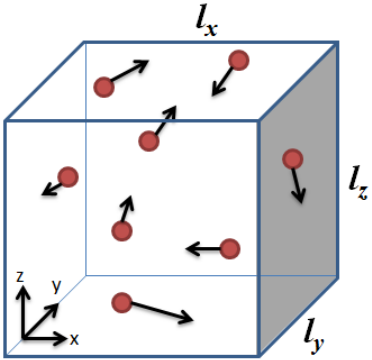
1. Special relativity
2. Mass-energy equivalence ($E = mc^2$)
3. Photoelectric effect
   $\rightarrow$ introduce the concept of energy quanta
   $\rightarrow$ wave-particle duality
   $\rightarrow$ the creation of quantum mechanics
4. Brownian motion
   $\rightarrow$ the foundation of stochastic finance

Brownian motion



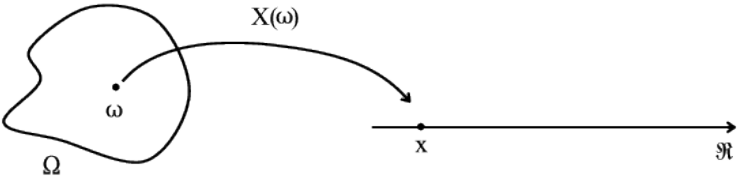
Brownian motion: the random motion of particles suspended in a fluid (a liquid or a gas) resulting from their collision with the fast-moving molecules in the fluid

The sample space of Brownian motion



- Experiment: let a pollen grain move in water, starting at a given position, for 1s
- Einstein's question: the trajectory of the pollen grain
- An outcome: a record of the locations and velocities of all the molecules in the system over time
- The sample space: the set of all possible outcomes

Random variable



Einstein's approach

- Describes the increment of particle positions in unrestricted one dimensional domain as a random variable Δ
- Computes the probability law Δ

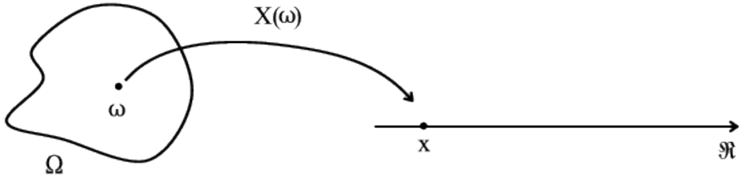
$$\rho(x,t) = \frac{N}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}.$$

$\rightarrow$ a Gaussian random variable

- enables the experimental determination of Avogadro's number and the size of molecules

The trajectory of the pollen grain

- $X_t(\omega)$ (or X_t), the trajectory of the pollen grain is a Brownian motion
- the existence and continuity of Brownian motion is established later by the Kolmogorov continuity theorem and the Kolmogorov extension theorem
  $\rightarrow$ creates the foundation for stochastic finance



Not only does God play dice, but... he sometimes throws them where they cannot be seen

—Stephen Hawking