Probability Theory and Simulation Methods

♥ Feb 14th, 2018 ♥

Lecture 5: Conditional probability

Topics

Week 1	Chapter 1: Axioms of probability
Week 2	**Chapter 3: Conditional probability and independence**
Week 3	Chapters 4,5,6,7: Random variables
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Overview

1. Conditional Probability
2. Law of Multiplication
3. Law of Total Probability
4. Bayes Formula
5. Independence

Sample space and events

1. An experiment: is any action, process, or phenomenon whose outcome is subject to uncertainty
2. An outcome: is a result of an experiment
   Each run of the experiment results in one outcome
3. A sample space: is the set of all possible outcomes of an experiment
4. An event: is a subset of the sample space.
   An event occurs when one of the outcomes that belong to it occurs

Axioms of probability

Definition (Probability Axioms) *Let S be the sample space of a random phenomenon. Suppose that to each event A of S, a number denoted by $P(A)$ is associated with A. If P satisfies the following axioms, then it is called a* ***probability*** *and the number $P(A)$ is said to be the* ***probability of A****.*

Axiom 1 $P(A) \geq 0$.

Axiom 2 $P(S) = 1$.

Axiom 3 *If $\{A_1, A_2, A_3, \dots\}$ is a sequence of mutually exclusive events (i.e., the joint occurrence of every pair of them is impossible: $A_i A_j = \emptyset$ when $i \neq j$), then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Some basic properties

Lemma

- *The probability of the empty set is* 0.
- *Let* A, B *be two disjoint events (that is* $AB = \emptyset$*). Then*

$$P[A \cup B] = P[A] + P[B]$$

- *Let* $A \subset B$ *, then* $P[A] \leq P[B]$
- *Let* $A \subset B$ *, then* $P[B] = P[A] + P[B \setminus A]$
- *Let* A^c *be the complement event of* A*. Then*

$$P[A^c] = 1 - P[A]$$

- *For any events* A, B

$$P[A] = P[AB] + P[AB^c]$$

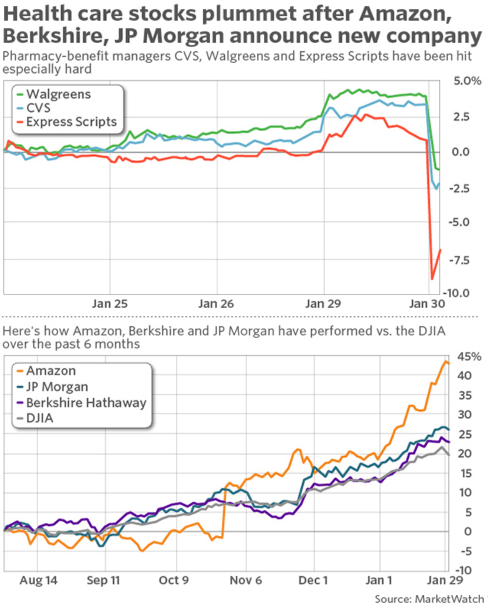
Probabilities

- Given an experiment and a sample space, we can define many probabilities
- Experiment: tossing a coin, $\mathcal{S} = \{H, T\}$
- If you believe the coin is fair:

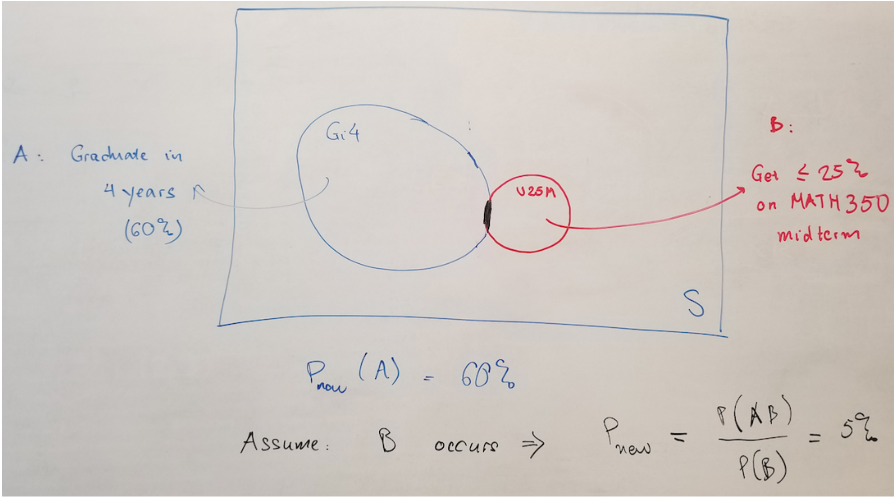
$$P(\emptyset) = 0, \qquad P(\{H\}) = 0.5, \qquad P(\{T\}) = 0.5, \qquad P(\{H, T\}) = 1$$

- If you don't believe so, maybe

$$Q(\emptyset) = 0, \qquad Q(\{H\}) = 0.6, \qquad Q(\{T\}) = 0.4, \qquad Q(\{H, T\}) = 1$$


Health care stocks plummet after Amazon, Berkshire, JP Morgan announce new company
Pharmacy-benefit managers CVS, Walgreens and Express Scripts have been hit especially hard
Walgreens
CVS
Express Scripts
5.0%
2.5
0.0
-2.5
-5.0
-7.5
-10.0
Jan 25
Jan 26
Jan 29
Jan 30
Here's how Amazon, Berkshire and JP Morgan have performed vs. the DJIA over the past 6 months
Amazon
JP Morgan
Berkshire Hathaway
DJIA
45%
40
35
30
25
20
15
10
5
0
-5
-10
Aug 14
Sep 11
Oct 9
Nov 6
Dec 1
Jan 1
Jan 29
Source: MarketWatch

Conditional probability



Conditional probability

Definition

Let $P(B) > 0$, the conditional probability of A given B, denoted by $P(A|B)$, is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Note: the equation is is neither an axiom nor a theorem. It is a definition.

About the notation

- From a mathematical point of view,

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\to x^2 \end{aligned}$$

  and

$$\begin{aligned} g : [0,1] &\to \mathbb{R} \\ x &\to x^2 \end{aligned}$$

  are two different functions
- To highlight the relation between them, mathematicians use the notation

$$f|_{[0,1]}$$

  to describe the second function

About the notation

- Similarly, the probability at the beginning of the semester

$$\begin{aligned} P : \{\text{set of all events on } \mathcal{S}\} &\to \mathbb{R} \\ A &\to P(A) \end{aligned}$$

and the probability after knowing B occurs

$$\begin{aligned} P_{new} : \{\text{set of all events on } \mathcal{S}\} &\to \mathbb{R} \\ A &\to P_{new}(A) = \frac{P(AB)}{P(B)} \end{aligned}$$

are two different probabilities

- To highlight the relation between them, we use the notation

$$P(A|B)$$

to describe $P_{new}(A)$

Conditional probability

Theorem 3.1 *Let S be the sample space of an experiment, and let B be an event of S with $P(B) > 0$. Then*

(a) $P(A \mid B) \geq 0$ *for any event A of S.*

(b) $P(S \mid B) = 1$.

(c) *If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then*

$$P\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \sum_{i=1}^{\infty} P(A_i \mid B).$$

Conditional probability

1. $P(\emptyset \mid B) = 0$.
2. $P(A^c \mid B) = 1 - P(A \mid B)$.
3. If $C \subseteq A$, then $P(AC^c \mid B) = P(A - C \mid B) = P(A \mid B) - P(C \mid B)$.
4. If $C \subseteq A$, then $P(C \mid B) \leq P(A \mid B)$.
5. $P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) - P(AC \mid B)$.
6. $P(A \mid B) = P(AC \mid B) + P(AC^c \mid B)$.

7. To calculate $P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \mid B)$, we calculate conditional probabilities of all possible intersections of events from $\{A_1, A_2, \ldots, A_n\}$, given B, add the conditional probabilities obtained by intersecting an odd number of the events, and subtract the conditional probabilities obtained by intersecting an even number of events.
8. For any increasing or decreasing sequences of events $\{A_n, n \geq 1\}$,

$$\lim_{n\to\infty} P(A_n \mid B) = P(\lim_{n\to\infty} A_n \mid B).$$

Problem

Problem

Let A, B *be two events such that* $P(A), P(B), P(B^c) > 0$*, show that*

(a)

$$P(AB) = P(B)P(A|B)$$

(b)

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

(c)

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Laws about conditional probabilities

S 3.2 Law of multiplication

$$P(AB) = P(B)P(A|B)$$

S 3.3 Law of total probability

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

S 3.4 Bayes' formula

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$