Probability Theory and Simulation Methods

Feb 19th, 2018

Lecture 7: Conditional probability: examples

Topics

Week 1	Chapter 1: Axioms of probability
Week 2	**Chapter 3: Conditional probability and independence**
Week 4	Chapters 4,5,6,7: Random variables
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Overview

1. Conditional Probability
2. Law of Multiplication
3. Law of Total Probability
4. Bayes Formula
5. Independence

Conditional probability

Definition

Let $P(B) > 0$, the conditional probability of A given B, denoted by $P(A|B)$, is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Laws about conditional probabilities

S 3.2 Law of multiplication

$$P(AB) = P(B)P(A|B)$$

S 3.3 Law of total probability

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

S 3.4 Bayes' formula

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Law of multiplication

Law of multiplication

- For two events
$$P(AB) = P(B)P(A|B)$$
- For three events
$$P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$$
- For n events
$$P(A_1A_2A_3 \dots A_{n-1}A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\dots P(A_n|A_1A_2\dots A_{n-1})$$

Law of multiplication: example

$$P(A_1 A_2 A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 A_2)$$

Problem

A consulting firm is awarded 43% of the contracts it bids on. Suppose that Nordulf works for a division of the firm that gets to do 15% of the projects contracted for. If Nordulf directs 35% of the projects submitted to his division, what percentage of all bids submitted by the firm will result in contracts for projects directed by Nordulf?

Law of multiplication: example

Problem

In a trial, the judge is 65% sure that Susan has committed a crime. Robert is a witness who knows whether Susan is innocent or guilty. However, Robert is Susans friend and will lie with probability 0.25 if Susan is guilty. He will tell the truth if she is innocent. What is the probability that Robert will commit perjury?

Law of total probability

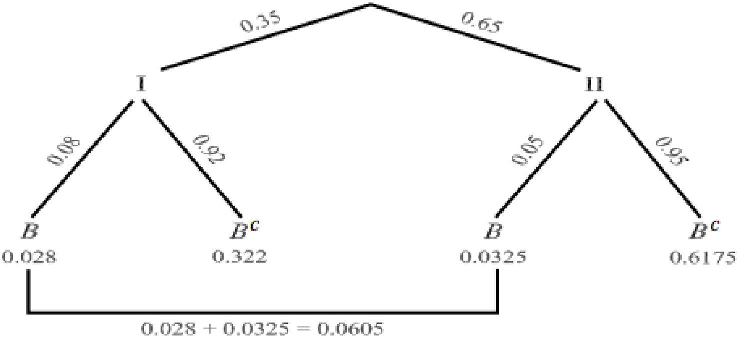
Example 1

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Example 1



0.35
0.65
I
II
0.08
0.92
0.05
0.95
B
B^c
B
B^c
0.028
0.322
0.0325
0.6175
0.028 + 0.0325 = 0.0605

Partition of the sample space

Definition *Let* $\{B_1, B_2, \ldots, B_n\}$ *be a set of nonempty subsets of the sample space* S *of an experiment. If the events* $B_1, B_2, \ldots, B_n$ *are mutually exclusive and* $\bigcup_{i=1}^{n} B_i = S$*, the set* $\{B_1, B_2, \ldots, B_n\}$ *is called a* ***partition*** *of* S.

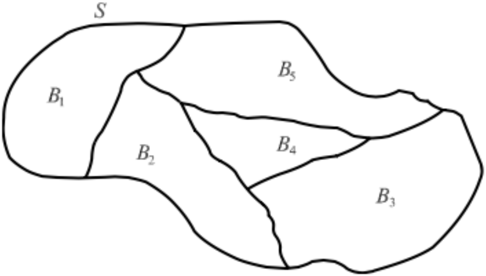


Figure 3.3 Partition of the given sample space S.

Law of total probability

Theorem 3.4 (Law of Total Probability) *If $\{B_1, B_2, \ldots, B_n\}$ is a partition of the sample space of an experiment and $P(B_i) > 0$ for $i = 1, 2, \ldots, n$, then for any event A of S,*

$$\begin{aligned} P(A) &= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n) \\ &= \sum_{i=1}^{n} P(A \mid B_i)P(B_i). \end{aligned}$$

More generally, let $\{B_1, B_2, \ldots\}$ be a sequence of mutually exclusive events of S such that $\bigcup_{i=1}^{\infty} B_i = S$. Suppose that, for all $i \geq 1$, $P(B_i) > 0$. Then for any event A of S,

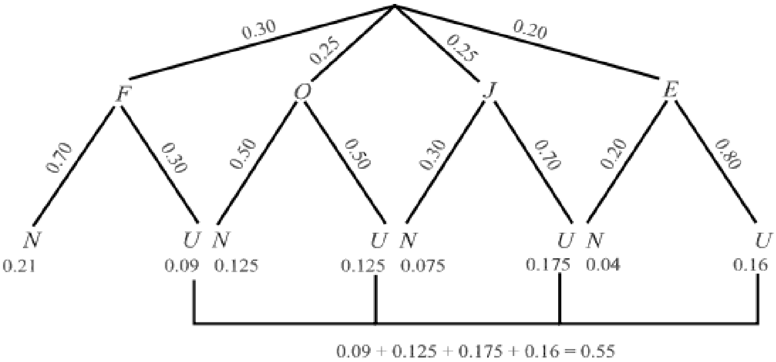
$$P(A) = \sum_{i=1}^{\infty} P(A \mid B_i)P(B_i).$$

Example 2

Problem

Suppose that 80% of the seniors, 70% of the juniors, 50% of the sophomores, and 30% of the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are seniors, what percent of all students use the library frequently?

Example 2


0.30
0.25
0.25
0.20
F
O
J
E
0.70
0.30
0.50
0.50
0.30
0.70
0.20
0.80
N
U
N
U
N
U
N
U
0.21
0.09
0.125
0.125
0.075
0.175
0.04
0.16
0.09 + 0.125 + 0.175 + 0.16 = 0.55

Bayes' formula

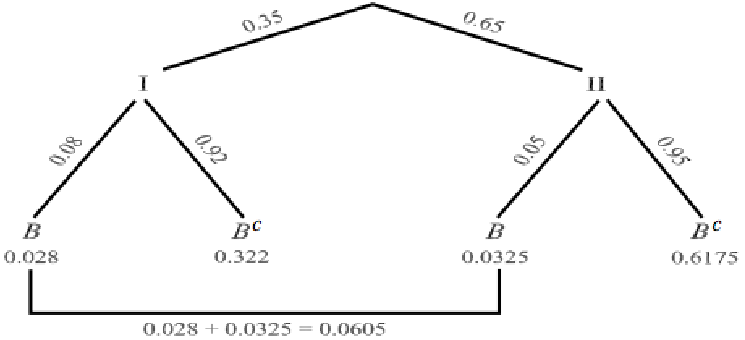
Example 1

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Question: Assume that a random selected car broke down, what is the probability that this car is from agency I?

Example 1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0.028 + 0.0325 = 0.0605

Example 2

Problem

Suppose that 80% of the seniors, 70% of the juniors, 50% of the sophomores, and 30% of the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are seniors, what percent of all students use the library frequently?

Question: You randomly run into a student at the library, what is the probability that they are a freshman?