Probability Theory and Simulation Methods

Feb 26th, 2018

Lecture 9: Random variables

Countdown to midterm (March 21st): 30 days

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 5, 6, 7: Random variables**
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Chapter 3: Review

1. Conditional Probability
2. Law of Multiplication
3. Law of Total Probability
4. Bayes' Formula
5. Independence

Conditional probability

Definition

Let $P(B) > 0$, the conditional probability of A given B, denoted by $P(A|B)$, is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Laws about conditional probabilities

S 3.2 Law of multiplication

$$P(AB) = P(B)P(A|B)$$

S 3.3 Law of total probability

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

S 3.4 Bayes' formula

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Independence

Definition

Two events A and B are called **independent** if:

$$P(AB) = P(A)P(B).$$

If two events are not independent, they are called dependent.

If A and B are independent, we say that $\{A, B\}$ is an independent set of events.

Remark: If A and B are independent then

$$P(A|B) = P(A), \quad P(B|A) = P(B)$$

Properties

Theorem

If A and B are independent, then A and B^c are independent as well.

Corollary: If A and B are independent, then A^c and B^c are independent as well.

Theorem

If A and B are mutually exclusive events and $P(A) > 0, P(B) > 0$, then they are dependent.

Three independent events

Definition

We say that {A, B, C} is an independent set of events if all of the following is true

$$P(AB) = P(A)P(B)$$
$$P(AC) = P(A)P(C)$$
$$P(BC) = P(B)P(C)$$
$$P(ABC) = P(A)P(B)P(C)$$

Example

Example

We throw two dice. Let A be the event “the sum of the points is 7”, B the event “die #1 came up 3”, and C the event “die #2 came up 4”.

Prove that

- A and B are independent
- A and C are independent
- B and C are independent
- $\{A, B, C\}$ is not an independent set of evetns

Hint: note that

$$A \cap B = B \cap C = C \cap A = \{(3, 4)\}$$

Independent events

Definition

The set of events $\{A_1, A_2, \ldots, A_n\}$ is called independent if for every subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$, $k \geq 2$, of $\{A_1, A_2, \ldots, A_n\}$,

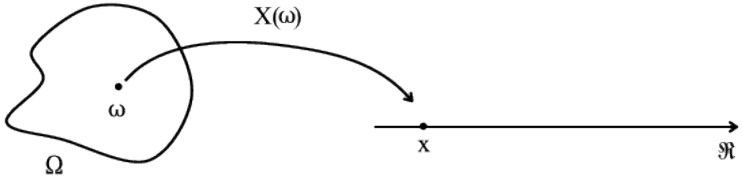
$$P(A_{i_1} A_{i_2}, \ldots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \ldots P(A_{i_k}).$$

Chapters 4, 5, 6, 7: Random variables

Order

- Discrete random variables (Chap 4)
- Continuous random variables (Chap 6)
- Special discrete distributions (Chap 5)
- Special continuous distributions (Chap 7)

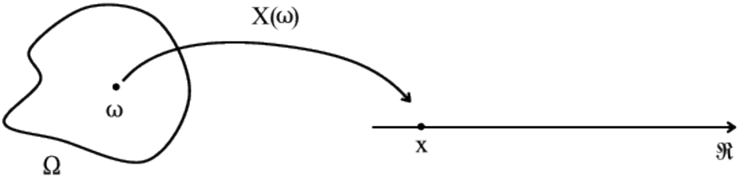
Random variable



Definition

Let S be the sample space of an experiment. A real-valued function $X : S \to \mathbb{R}$ is called a random variable of the experiment.

Notations



- When we write $\{X = 2\}$, we are (implicitly) referring to the set

$$\{s \in S : X(s) = 2\}$$

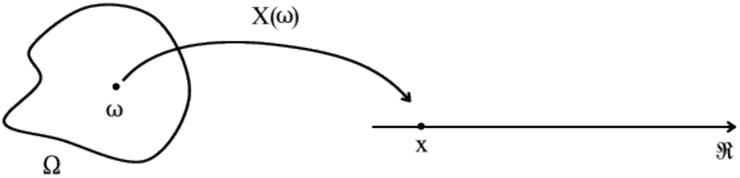
  which is an event of the experiment (a subset of S).
- When we write $\{X \in (0, 1)\}$, we are (implicitly) referring to the event

$$\{s \in S : X(s) \in (0, 1)\}.$$

Random variable: Example

- Experiment: Toss a fair coins 2 times
- Let X be the number of heads $\rightarrow X$ is a random variable
- What are all possible values of X?
- What is the probability that $X = 1$?

Random variable: Example



X	0	1	2
probability	0.25	0.5	0.25

Random variable: Example

- Experiment: You order a pizza
- Let X be The time elapsed, in minutes, between the placement of an order and its delivery $\rightarrow$ X is a random variable
- What are all possible values of X?