Probability Theory and Simulation Methods

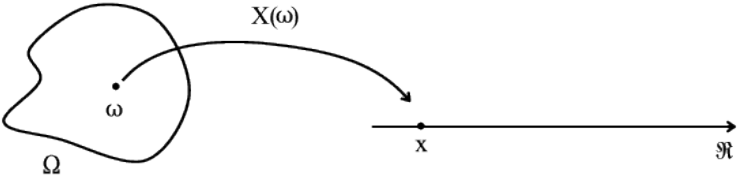
March 2nd, 2018

Lecture 11: Expectation

Countdown to midterm (March 21st): 26 days

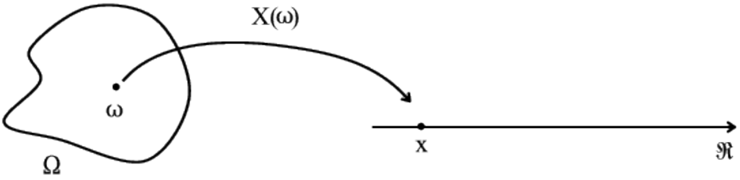
Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 5, 6, 7: Random variables**
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Random variable



Definition

Let S be the sample space of an experiment. A real-valued function $X : S \to \mathbb{R}$ is called a random variable of the experiment.



When we write $\{X \in (0,1)\}$, we are (implicitly) referring to the event

$$\{s \in S : X(s) \in (0,1)\}.$$

Discrete random variable

Definition

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

Note: A set A is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers, i.e, we can index the element of A as a sequence

$$A = \{x_1, x_2, \dots, x_n, \dots\}$$

A random variable X is described by its probability mass function

Definition *The* ***probability mass function*** *p of a random variable X whose set of possible values is $\{x_1, x_2, x_3, \dots\}$ is a function from $\mathbf{R}$ to $\mathbf{R}$ that satisfies the following properties.*

(a) $p(x) = 0$ *if* $x \notin \{x_1, x_2, x_3, \dots\}$.

(b) $p(x_i) = P(X = x_i)$ *and hence* $p(x_i) \geq 0$ $(i = 1, 2, 3, \dots)$.

(c) $\sum_{i=1}^{\infty} p(x_i) = 1$.

Read the pmf table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

Expectation

Definition *The **expected value** of a discrete random variable X with the set of possible values A and probability mass function $p(x)$ is defined by*

$$E(X) = \sum_{x \in A} x p(x).$$

We say that $E(X)$ exists if this sum converges absolutely.

The expected value of a random variable X is also called the **mean**, or the **mathematical expectation**, or simply the **expectation** of X. It is also occasionally denoted by $E[X]$, $E(X)$, EX, μ_X, or μ.

Exercise

Problem

A random variable X has the following pmf table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

What is the expected value of X?

Exercise

Problem

We write the numbers $a_1 = 2, a_2 = 4, \ldots, a_n = 2n$ *on* n *identical balls and mix them in a box. What is the expected value of the number on a ball selected at random?*

Exercise

Problem

We write the numbers $a_1 = 2, a_2 = 4, \ldots, a_n = 2n$ on n identical balls and mix them in a box. What is the expected value of the number on a ball selected at random?

X	2	4	6	...	2n
probability	$1/n$	1/n	1/n	...	1/n

St. Petersburg Paradox

Problem

Suppose that we flip a coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars.

- *Write down the probability mass function of the payment*
- *Compute its expected value*

Problem

Suppose that we flip a coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars.

- *Write down the probability mass function of the payment*
- *Compute its expected value*

X	2	2^2	2^3	$\ldots$	2^n	$\ldots$
probability	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\ldots$	$\frac{1}{2^n}$	$\ldots$

Function of a random variable

- A random variable X has the following pmf table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- Define $Y = X^2$, then Y is a random variable of the same experiment as X
- Write down the pmf table of Y
- What is the expected value of Y?

Key idea

Since $Y = X^2$, the two expression

$$\{s \in S : X(s) = 2\}$$

and

$$\{s \in S : Y(s) = 4\}$$

describes the same events.
Thus

$$P(X = 2) = P(X^2 = 4) = P(Y = 4)$$

Function of a random variable

- A random variable X has the following pmf table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- Define $Y = g(X)$, then Y is a random variable of the same experiment as X
- What is the expected value of Y?

Law of the unconscious statistician (LOTUS)

Theorem 4.2 *Let X be a discrete random variable with set of possible values A and probability mass function $p(x)$, and let g be a real-valued function. Then $g(X)$ is a random variable with*

$$E[g(X)] = \sum_{x \in A} g(x)p(x).$$

Corollary *Let X be a discrete random variable; $g_1, g_2, \ldots, g_n$ be real-valued functions, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$\begin{aligned} &E\big[\alpha_1 g_1(X) + \alpha_2 g_2(X) + \cdots + \alpha_n g_n(X)\big] \\ &\quad = \alpha_1 E\big[g_1(X)\big] + \alpha_2 E\big[g_2(X)\big] + \cdots + \alpha_n E\big[g_n(X)\big]. \end{aligned}$$

Example 4.23 The probability mass function of a discrete random variable X is given by

$$p(x) = \begin{cases} x/15 & x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of $X(6 - X)$?

Exercise

Problem

Show that if X is a constant random variable, that is, if $P(X = c) = 1$ for a constant c, then $E(X) = c$.

The probability mass function:

x	c
p(x)	1

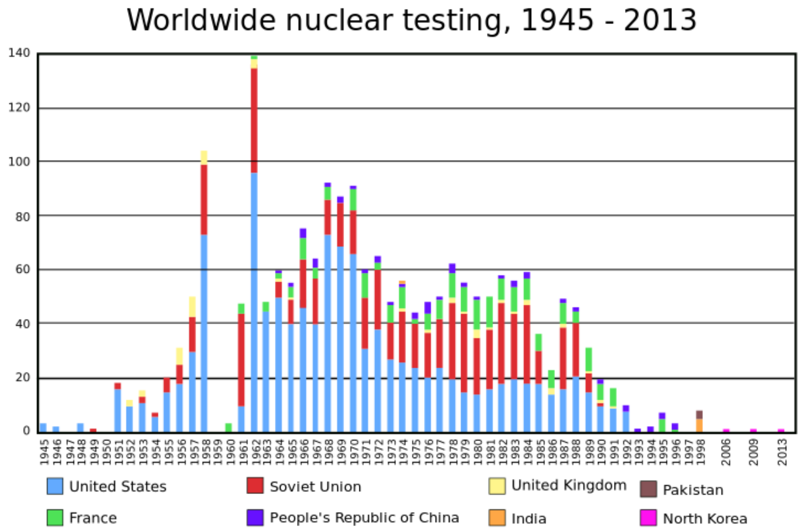
What we need to know so far

- What is a discrete random variable?
- What is a pmf?
- How to compute expected value of a random variable?
- How to compute the expectation of $g(X)$?

Genralization

- Inputs: $X_1, X_2, \ldots, X_n$
- Model: Function h
- Output $T = h(X_1, X_2, \ldots, X_n)$

Question: If we know the pmf of the inputs, can we estimate the pmf of the output?


Worldwide nuclear testing, 1945 - 2013
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