Probability Theory and Simulation Methods

March 7th, 2018

Lecture 13: Distribution functions

Countdown to midterm (March 21st): 14 days

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 5, 6, 7: Random variables**
Week 9	Chapters 8, 9: Bivariate and multivariate distributions
Week 10	Chapter 10: Expectations and variances
Week 11	Chapter 11: Limit theorems
Week 12	Chapters 12, 13: Selected topics

Chapter 4: Discrete random variables

4.1 Random variables

4.3 Discrete random variables

4.4 Expectations of discrete random variables

4.5 Variances and moments of discrete random variables

4.2 Distribution functions

Discrete random variable

Definition

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

Note: A set A is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers, i.e, we can index the element of A as a sequence

$$A = \{x_1, x_2, \ldots, x_n, \ldots\}$$

Discrete random variable

A random variable X is described by its probability mass function

Definition *The **probability mass function** p of a random variable X whose set of possible values is $\{x_1, x_2, x_3, \dots\}$ is a function from $\mathbf{R}$ to $\mathbf{R}$ that satisfies the following properties.*

(a) $p(x) = 0$ *if* $x \notin \{x_1, x_2, x_3, \dots\}$.

(b) $p(x_i) = P(X = x_i)$ *and hence* $p(x_i) \geq 0$ $(i = 1, 2, 3, \dots)$.

(c) $\sum_{i=1}^{\infty} p(x_i) = 1$.

Represent the probability mass function

- As a table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- As a function:

$$p(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 1, 2, 3, \ldots, \\ 0 & \text{elsewhere} \end{cases}$$

Expectation

Definition *The **expected value** of a discrete random variable X with the set of possible values A and probability mass function $p(x)$ is defined by*

$$E(X) = \sum_{x \in A} xp(x).$$

We say that $E(X)$ exists if this sum converges absolutely.

The expected value of a random variable X is also called the **mean**, or the **mathematical expectation**, or simply the **expectation** of X. It is also occasionally denoted by $E[X]$, $E(X)$, EX, μ_X, or μ.

Law of the unconscious statistician (LOTUS)

Theorem 4.2 *Let X be a discrete random variable with set of possible values A and probability mass function $p(x)$, and let g be a real-valued function. Then $g(X)$ is a random variable with*

$$E[g(X)] = \sum_{x \in A} g(x)p(x).$$

Examples:

$$E[X^2] = \sum_{x \in A} x^2 p(x)$$

$$E[X^7 - X - 1] = \sum_{x \in A} (x^7 - x - 1)p(x)$$

$$E[(X-2)^2] = \sum_{x \in A} (x-2)^2 p(x)$$

Law of the unconscious statistician (LOTUS)

Corollary *Let X be a discrete random variable; $g_1, g_2, \ldots, g_n$ be real-valued functions, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$E\left[\alpha_1 g_1(X) + \alpha_2 g_2(X) + \cdots + \alpha_n g_n(X)\right] = \alpha_1 E\left[g_1(X)\right] + \alpha_2 E\left[g_2(X)\right] + \cdots + \alpha_n E\left[g_n(X)\right].$$

Proof:

$$\begin{aligned}
&E[\alpha_1 g_1(X) + \alpha_2 g_2(X) + \ldots \alpha_n g_n(X)] \\
&= \sum_{x \in A} (\alpha_1 g_1(x) + \alpha_2 g_2(x) + \ldots \alpha_n g_n(x)) p(x) \\
&= \sum_{x \in A} \alpha_1 g_1(x) p(x) + \sum_{x \in A} \alpha_2 g_2(x) p(x) + \ldots \sum_{x \in A} \alpha_n g_n(x) p(x) \\
&= \alpha_1 E[g_1(X)] + \alpha_2 E[g_2(X)] + \ldots + \alpha_n E[g_n(X)]
\end{aligned}$$

Variance: definition

Definition

Let X be a discrete random variable with a set of possible values A, probability mass function p(x), and $E(X) = \mu$. Then Var(X) and σ_X, called the variance and the standard deviation of X, respectively, are defined by

$$Var(X) = E[(X - \mu)^2] \quad \text{and} \quad \sigma_X = \sqrt{Var(X)}$$

Variance: computing formula

Theorem

$$Var(X) = E(X^2) - (EX)^2$$

Properties

Theorem

Let X be a discrete random variable; then for constants a and b we have that

$$Var(aX + b) = a^2 Var(X)$$

Example

Example

Suppose that, for a discrete random variable X, $E(X) = 2$ and $E[X(X-4)] = 5$.
Find the variance and the standard deviation of

$$12 - 4X.$$

Moments

$E[g(X)]$	**Definition**
$E(X^n)$	The nth moment of X
$E(\|X\|^r)$	The rth absolute moment of X
$E(X-c)$	The first moment of X about c
$E[(X-c)^n]$	The nth moment of X about c
$E[(X-\mu)^n]$	The nth central moment of X

What we need to know so far

- What is a discrete random variable?
- What is a pmf?
- How to compute expected value of a random variable?
- How to compute the expectation of $g(X)$?
- How to compute the variance of X?

Distribution functions

Computing probabilities concerning a random variable

Computing probability

Given a random variable X, we want to compute

- $P[X = a]$, $P[X > a]$, $P[X \geq a]$
- $P[X < b]$, $P[X \leq b]$
- $P[a \leq X \leq b]$, $P[a < X \leq b]$, $P[a \leq X < b]$
- $P[a < X < b]$

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.

Example

Example

The random variable X is described by the following pmf table

x	0	2
p(x)	0.3	0.7

- Compute $P[X \leq 1.5]$, $P[X \leq 2]$, $P[X \leq -1]$, $P[X \leq 0]$
- What is the general formula for $P[X \leq t]$?

Computing probabilities concerning a random variable

1. To calculate $P(X > a)$, note that $P(X > a) = 1 - P(X \leq a)$, thus

$$P(X > a) = 1 - F(a).$$

2. To calculate $P(a < X \leq b)$, $b > a$, note that $\{a < X \leq b\} = \{X \leq b\} - \{X \leq a\}$ and $\{X \leq a\} \subseteq \{X \leq b\}$. Hence, by Theorem 1.5,

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a).$$

3. To calculate $P(X < a)$, note that the sequence of the events $\{X \leq a - 1/n\}$ is an increasing sequence that converges to $\bigcup_{n=1}^{\infty} \{X \leq a - 1/n\} = \{X < a\}$. Therefore, by the continuity property of the probability function (Theorem 1.8),

$$\lim_{n \to \infty} P\left(X \leq a - \frac{1}{n}\right) = P\left(\bigcup_{n=1}^{\infty} \left\{X \leq a - \frac{1}{n}\right\}\right) = P(X < a),$$

which means that

$$P(X < a) = \lim_{n \to \infty} F\left(a - \frac{1}{n}\right).$$

Hence $P(X < a)$ is the left-hand limit of the function F as $x \to a$; that is,

$$P(X < a) = F(a-).$$

Example

Example 4.7 The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{1}{12}x + \frac{1}{2} & 2 \le x < 3 \\ 1 & x \ge 3, \end{cases}$$

where the graph of F is shown in Figure 4.1. Compute the following quantities:

(a) $P(X < 2)$; **(b)** $P(X = 2)$; **(c)** $P(1 \le X < 3)$;
(d) $P(X > 3/2)$; **(e)** $P(X = 5/2)$; **(f)** $P(2 < X \le 7)$.