Probability Theory and Simulation Methods

March 12nd, 2018

Lecture 14: Continuous random variables

Countdown to midterm (March 21st): 9 days

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 6: Random variables**
Week 9	**Chapter 5, 7: Special distributions**
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Order

- Discrete random variables (Chap 4)
- Continuous random variables (Chap 6)
- Special discrete distributions (Chap 5)
- Special continuous distributions (Chap 7)

- What is a discrete random variable?
- What is a pmf?
- How to compute expected value of a random variable?
- How to compute the expectation of $g(X)$?
- How to compute the variance of X?
- Distribution function of X

Chapter 6: Continuous random variables

Discrete random variable: probability mass function

- A discrete r.v. is characterized by it probability mass function

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- $P(X = 3) = 0.13$

$$P\left(X \in [3.5, 5.5]\right) = \sum_{x_i \in [3.5, 5.5]} p(x_i) = 0.25 + 0.39 = 0.64$$

- For a continuous random variable, the set of all possible values are uncountably infinite

Continuous random variable

- Example: X is the waiting time for a pizza to be delivered
- In this example, the set of all possible values are uncountably infinite, and

$$P(X = 25) = 0,$$

so the expression $P(X = 25)$ does not convey any information
- We can still talk about $P\left(X \in [20, 30]\right)$ but the quantity

$$\sum_{x \in [20,30]} p(x)$$

does not make sense

Continuous random variable

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **absolutely continuous** or, for simplicity, **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

Whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies the equation above.

Properties

Let X be a continuous r.v. with density function f, then

- $\int_{-\infty}^{\infty} f(x)\,dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$P(X = a) = 0$$

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$$

Probability density function

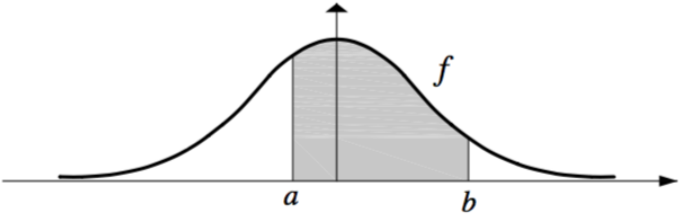

Figure 6.1 The shaded area under f is the probability that $X \in I = (a, b)$.

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} ce^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is some unknown constant.

- *Compute c*
- *Compite $P(X \in [1,2])$*

Expectation

Definition *If X is a continuous random variable with probability density function f, the **expected value** of X is defined by*

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

The expected value of X is also called the **mean**, or **mathematical expectation**, or simply the **expectation** of X, and as in the discrete case, sometimes it is denoted by EX, $E[X]$, μ, or μ_X.

Lotus

Theorem 6.3 *Let X be a continuous random variable with probability density function $f(x)$; then for any function $h: \mathbf{R} \to \mathbf{R}$,*

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx.$$

Linearity of expectation

Corollary *Let X be a discrete random variable; $g_1, g_2, \dots, g_n$ be real-valued functions, and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be real numbers. Then*

$$E\big[\alpha_1 g_1(X) + \alpha_2 g_2(X) + \cdots + \alpha_n g_n(X)\big] \\ = \alpha_1 E\big[g_1(X)\big] + \alpha_2 E\big[g_2(X)\big] + \cdots + \alpha_n E\big[g_n(X)\big].$$

Proof:

$$\begin{aligned} & E[\alpha_1 g_1(X) + \alpha_2 g_2(X) + \ldots \alpha_n g_n(X)] \\ &= \sum_{x \in A} (\alpha_1 g_1(x) + \alpha_2 g_2(x) + \ldots \alpha_n g_n(x)) p(x) \\ &= \sum_{x \in A} \alpha_1 g_1(x) p(x) + \sum_{x \in A} \alpha_2 g_2(x) p(x) + \ldots \sum_{x \in A} \alpha_n g_n(x) p(x) \\ &= \alpha_1 E[g_1(X)] + \alpha_2 E[g_2(X)] + \ldots + \alpha_n E[g_n(X)] \end{aligned}$$

Corollary *Let X be a continuous random variable with probability density function $f(x)$. Let $h_1, h_2, \ldots, h_n$ be real-valued functions, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$E\left[\alpha_1 h_1(X) + \alpha_2 h_2(X) + \cdots + \alpha_n h_n(X)\right] \\ = \alpha_1 E\left[h_1(X)\right] + \alpha_2 E\left[h_2(X)\right] + \cdots + \alpha_n E\left[h_n(X)\right].$$

Proof:

$$E[\alpha_1 h_1(X) + \alpha_2 h_2(X) + \ldots \alpha_n h_n(X)]$$

$$= \int (\alpha_1 h_1(x) + \alpha_2 h_2(x) + \ldots \alpha_n h_n(x)) f(x)$$

$$= \int \alpha_1 h_1(x) f(x) + \int \alpha_2 h_2(x) f(x) + \ldots \int \alpha_n h_n(x) f(x)$$

$$= \alpha_1 E[g_1(X)] + \alpha_2 E[g_2(X)] + \ldots + \alpha_n E[g_n(X)]$$

Variance

Definition *If X is a continuous random variable with* $E(X) = \mu$*, then* $\mathrm{Var}(X)$ *and* σ_X*, called the* ***variance*** *and* ***standard deviation*** *of X, respectively, are defined by*

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right],$$

$$\sigma_X = \sqrt{E\left[(X-\mu)^2\right]}.$$

We also have

$$Var(X) = E(X^2) - (EX)^2$$

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} ce^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is some unknown constant.
Compute $E[X]$ and $Var(X)$.

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.