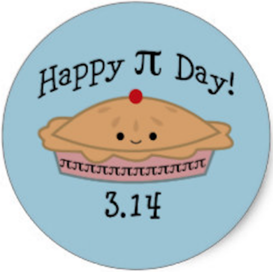
Probability Theory and Simulation Methods



Lecture 15: Continuous random variables
—Expectations and Variances—

Countdown to midterm (March 21st): 7 days

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 6: Random variables**
Week 9	**Chapter 5, 7: Special distributions**
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Chapter 6: Continuous random variables

Continuous random variable

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **absolutely continuous** or, for simplicity, **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

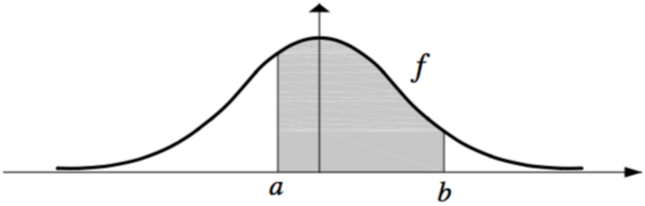
Whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies the equation above.

Properties

Let X be a continuous r.v. with density function f, then

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$



Expectation

Definition *If X is a continuous random variable with probability density function f, the* ***expected value*** *of X is defined by*

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

The expected value of X is also called the **mean**, or **mathematical expectation**, or simply the **expectation** of X, and as in the discrete case, sometimes it is denoted by EX, $E[X]$, μ, or μ_X.

Theorem 6.3 *Let X be a continuous random variable with probability density function $f(x)$; then for any function $h: \mathbf{R} \to \mathbf{R}$,*

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx.$$

Corollary *Let X be a continuous random variable with probability density function $f(x)$. Let $h_1, h_2, \ldots, h_n$ be real-valued functions, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. Then*

$$\begin{aligned} &E[\alpha_1 h_1(X) + \alpha_2 h_2(X) + \cdots + \alpha_n h_n(X)] \\ &\quad = \alpha_1 E[h_1(X)] + \alpha_2 E[h_2(X)] + \cdots + \alpha_n E[h_n(X)]. \end{aligned}$$

Variance

Definition *If X is a continuous random variable with $E(X) = \mu$, then* $\mathrm{Var}(X)$ *and* σ_X*, called the* ***variance*** *and* ***standard deviation*** *of X, respectively, are defined by*

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right],$$

$$\sigma_X = \sqrt{E\left[(X-\mu)^2\right]}.$$

We also have

$$Var(X) = E(X^2) - (EX)^2$$

Example 1

Problem

The time it takes for a student to finish an aptitude test (in hours) has the density function

$$f(x) = \begin{cases} c(x-1)(2-x) & \text{if } 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where c is some unknown constant.

- *Compute c*
- *Compute $P(X \in [1, 2])$*

Example 1 (cont.)

Problem

The time it takes for a student to finish an aptitude test (in hours) has the density function

$$f(x) = \begin{cases} c(x-1)(2-x) & \text{if } 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where c is some unknown constant.
Determine the mean and standard deviation of the time it takes for a randomly selected student to finish the aptitude test.

Example 2

Problem

A random variable X has the density function

$$f(x) = \begin{cases} ce^{-3x} & \text{if } 1 \leq 0 \leq x \\ 0 & \text{elsewhere} \end{cases}$$

where c is some unknown constant.

- *Compute c*
- *Compute* $E[e^X]$

Example 3

Problem

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{c}{x^2} & \text{if } 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

where c is some unknown constant.

- *Compute c*
- *Compute $E[\ln(X)]$*
- *Compute $E[X]$ and σ_X*

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.

Distribution function

For continuous random variable:

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

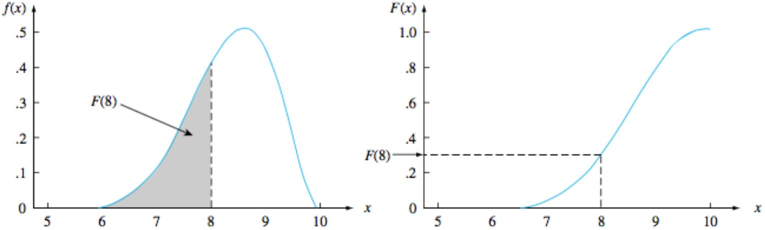
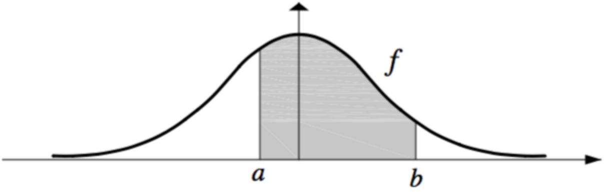


Figure 4.5 A pdf and associated cdf

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



Moreover:

$$f(x) = F'(x)$$

Example 4

Problem

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(x) = \begin{cases} 1 - \frac{16}{x^2} & \text{if } x \geq 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E[X]$.