Probability Theory and Simulation Methods

March 23st, 2018

Lecture 17: Continuous random variables

—Density function of a function of a random variable—

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 6: Random variables**
Week 9	**Chapter 5, 7: Special distributions**
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Chapter 6: Continuous random variables

Continuous random variable

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **absolutely continuous** or, for simplicity, **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

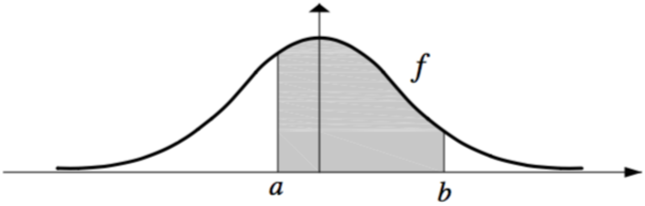
Whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies the equation above.

Properties

Let X be a continuous r.v. with density function f, then

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$



Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.

For continuous random variable:

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

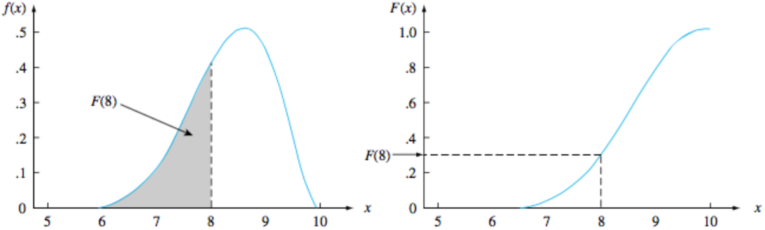
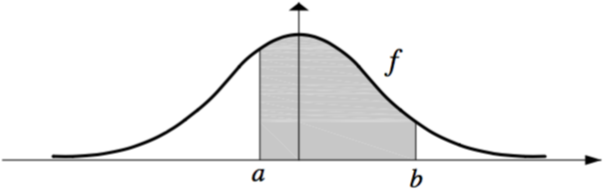


Figure 4.5 A pdf and associated cdf

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



Moreover:

$$f(x) = F'(x)$$

Example 1

Problem

A random variable X has the density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Write down the formula for the distribution function $F_X(t)$

Example 2

Problem

A random variable X has the density function

$$f(x) = \begin{cases} \frac{x^3}{4} & \text{if } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Write down the formula for the distribution function $F_X(t)$

Example 3

Problem

The distribution function of a continuous random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x+1}{2} & \text{if } -1 \leq x \leq 1 \\ 1 & \text{elsewhere} \end{cases}$$

- *What is $P[X \geq 1/2]$*
- *Compute $E(X)$*

Example 4

Problem

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(y) = \begin{cases} 1 - \frac{16}{y^2} & \text{if } y \geq 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E[Y]$.

Function of a random variable: example

Problem

Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density functions of $Y = X^3$.

Idea:

- Fixed t, write down $F_Y(y) = P(Y \leq y)$ and represent it in term of f
- Take the derivative with respect to y to get the density function

Example 1

Recall that

$$F_X(t) = P(X \leq t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 2 - \frac{2}{t} & \text{if } 1 < t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

Then

$$F_Y(y) = P[Y \leq y] = P[X^3 \leq y] = P[X \leq y^{1/3}]$$

$$= \begin{cases} 0 & \text{if } y^{1/3} \leq 1 \\ 2 - \frac{2}{y^{1/3}} & \text{if } 1 < y^{1/3} < 2 \\ 1 & \text{if } y^{1/3} \geq 2 \end{cases}$$

Example 1

We have

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq 1 \\ 2 - \frac{2}{y^{1/3}} & \text{if } 1 < y < 8 \\ 1 & \text{if } y \geq 8 \end{cases}$$

Take derivative with respect to y

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{2}{3}\frac{1}{y^{4/3}} & \text{if } 1 < y < 8 \\ 0 & \text{elsewhere} \end{cases}$$

Example 2

Problem

Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density functions of $Y = \sqrt{X}$.