Probability Theory and Simulation Methods

April 2nd, 2018

Lecture 18: Special distributions

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	**Chapters 4, 6: Random variables**
Week 9	**Chapter 5, 7: Special distributions**
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Function of a random variable

For continuous random variable:

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

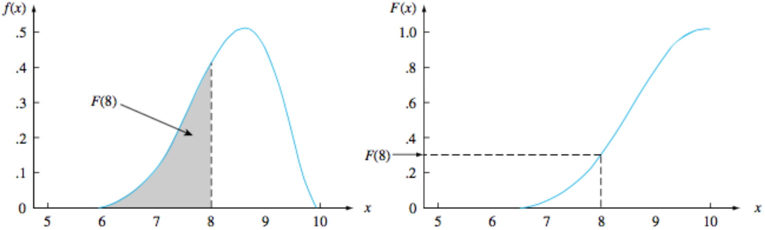
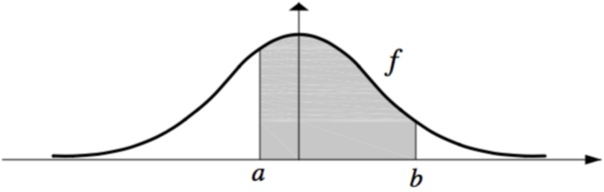


Figure 4.5 A pdf and associated cdf

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



Moreover:

$$f(x) = F'(x)$$

Function of a random variable: example

Problem

Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density functions of $Y = X^3$.

Idea:

- Fixed t, write down $F_Y(y) = P(Y \leq y)$ and represent it in term of f
- Take the derivative with respect to y to get the density function

Example 1

Recall that

$$F_X(t) = P(X \leq t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 2 - \frac{2}{t} & \text{if } 1 < t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

Then

$$F_Y(y) = P[Y \leq y] = P[X^3 \leq y] = P[X \leq y^{1/3}]$$

$$= \begin{cases} 0 & \text{if } y^{1/3} \leq 1 \\ 2 - \frac{2}{y^{1/3}} & \text{if } 1 < y^{1/3} < 2 \\ 1 & \text{if } y^{1/3} \geq 2 \end{cases}$$

Example 1

We have

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq 1 \\ 2 - \frac{2}{y^{1/3}} & \text{if } 1 < y < 8 \\ 1 & \text{if } y \geq 8 \end{cases}$$

Take derivative with respect to y

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{2}{3}\frac{1}{y^{4/3}} & \text{if } 1 < y < 8 \\ 0 & \text{elsewhere} \end{cases}$$

Example 2

Problem

Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density functions of $Y = \sqrt{X}$.

Bernoulli and Binomial random variables

Bernoulli trial

- Two outcomes: one outcome is usually called a success, denoted by s; the other outcome is called a failure, denoted by f
- Sample space: $\{s, f\}$
- The random variable defined by $X(s) = 1$ and $X(f) = 0$ is called a Bernoulli random variable
- The pmf of a Bernoulli random variable is

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{elsewhere} \end{cases}$$

where p is a parameter, referred to as the probability of a success

Problem

Consider a Bernoulli random variable X with pmf

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{elsewhere} \end{cases}$$

Compute $E(X)$ and $Var(X)$ as functions of p.

Independent events

Definition

The set of events $\{A_1, A_2, \dots, A_n\}$ is called independent if for every subset $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$, $k \geq 2$, of $\{A_1, A_2, \dots, A_n\}$,

$$P(A_{i_1} A_{i_2}, \dots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_k}).$$

A sequence of independent Bernoulli trials

Definition

Let $X_1, X_2, \ldots, X_n$ be a sequence of Bernoulli random variables. If, for all $j_i \in \{0, 1\}$, the sequence of events

$$\{X_1 = j_1\}, \{X_2 = j_2\}, \ldots, \{X_n = j_n\}$$

are independent, we say that $\{X_1, X_2, \ldots, X_n\}$ are independent.

Binomial random variables

Definition

If n Bernoulli trials all with probability of success p are performed independently, then X, the number of successes is called a binomial random variable with parameters n and p.

- The set of possible values of X is $\{0, 1, 2, ..., n\}$
- What is the pmf of X?

Definition *An unordered arrangement of r objects from a set A containing n objects $(r \leq n)$ is called an* ***r-element combination*** *of A, or a combination of the elements of A taken r at a time.*

Notation: By the symbol $\binom{n}{r}$ (read: n choose r) we mean the number of all r-element combinations of n objects. Therefore, for $r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

pmf of binomial random variables

Theorem 5.1 *Let X be a binomial random variable with parameters n and p. Then $p(x)$, the probability mass function of X, is*

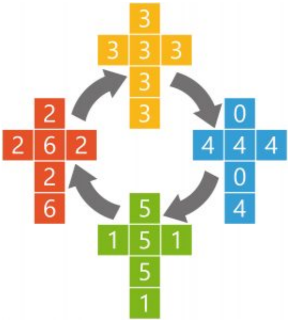
$$p(x) = P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x = 0, 1, 2, \ldots, n \\ 0 & \text{elsewhere.} \end{cases} \tag{5.2}$$

Example 1

Example

A restaurant serves 8 entrees of fish, 12 of beef, and 10 of poultry. If customers select from these entrees randomly, what is the probability that two of the next four customers order fish entrees?

Example 2: non-transitive die



- Suppose you are on the orange team, and you have to play with the blue team
- Three rounds of dice are played
- What is the probability that you win exactly 2 rounds?