Probability Theory and Simulation Methods

April 11st, 2018

Lecture 21: Special continuous distributions

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	Chapters 4, 6: Random variables
Week 9	**Chapter 5, 7: Special distributions**
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Special continuous distributions

Chapter 6: Special continuous distributions

- Uniform random variables
- Normal random variables
- Exponential random variables
- Other discrete random variables
  - Gamma distribution
  - Beta distributions

For continuous random variable:

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

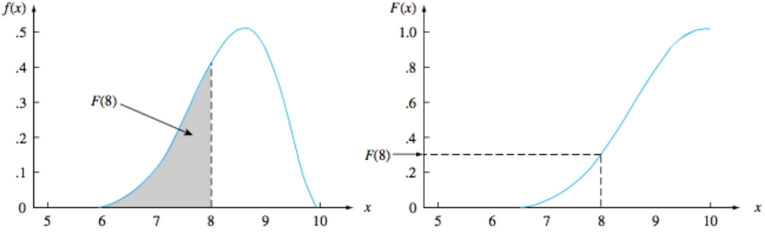
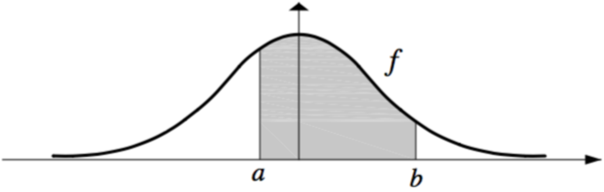


Figure 4.5 A pdf and associated cdf

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$



Moreover:

$$f(x) = F'(x)$$

Uniform random variables

$$F(t) = \begin{cases} 0 & t < a \\ \dfrac{t-a}{b-a} & a \le t < b \\ 1 & t \ge b. \end{cases}$$

$$f(t) = F'(t) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < t < b \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

Definition *A random variable X is said to be* ***uniformly distributed*** *over an interval (a, b) if its probability density function is given by* (7.1).

If X is uniformly distributed over an interval (a, b), then

$$E(X) = \frac{a+b}{2}, \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}, \qquad \sigma_X = \frac{b-a}{\sqrt{12}}.$$

Normal random variables

Definition *A random variable X is called* ***normal****, with parameters μ and σ, if its density function is given by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty.$$

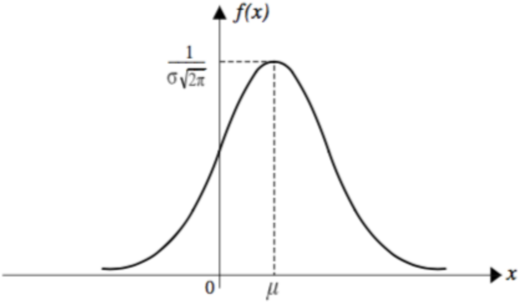
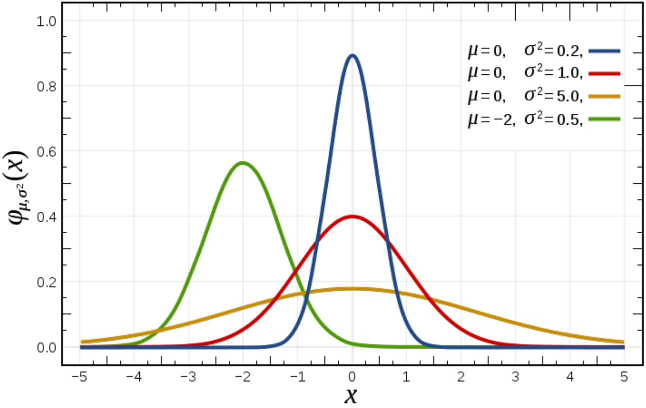


Figure 7.8 Density of $N(\mu, \sigma^2)$.

$\mathcal{N}(\mu, \sigma^2)$



$$E(X) = \mu, Var(X) = \sigma^2$$

Standard normal distribution

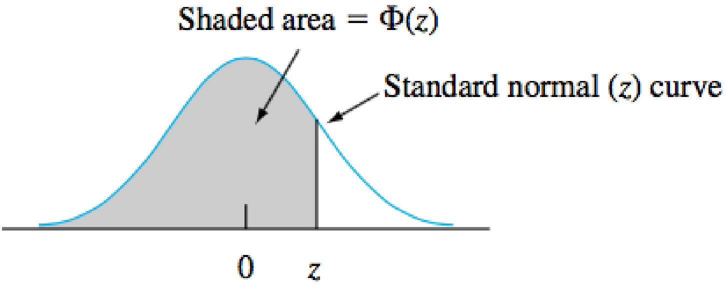
- If Z is a normal random variable with parameters $\mu = 0$ and $\sigma = 1$, then the pdf of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

and Z is called the standard normal distribution

- $E(Z) = 0$, $Var(Z) = 1$
- The cumulative distribution function of the standard normal distribution is:

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(y)\, dy$$



$$\Phi(t) = P(Z \le t) = \int_{-\infty}^{z} f(y)\, dy$$

$\Phi(z)$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Shifting and scaling normal random variables

Problem

Let X be a normal random variable with mean μ and standard deviation σ.
Then

$$Z = \frac{X - \mu}{\sigma}$$

follows the standard normal distribution.

Shifting and scaling normal random variables

If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution. Thus

$$\begin{aligned} P(a \le X \le b) &= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

$$P(X \le a) = \Phi\left(\frac{a-\mu}{\sigma}\right) \quad P(X \ge b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$$

Example 1

Example

Let X be a normal random variable with mean 39.8 and standard deviation 2.05. Compute

- $P[X \leq 40]$
- $P[X \geq 40]$
- $P[39.8 \leq X \leq 40]$

$\Phi(z)$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \le z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Example 2

Problem

Let X, the grade of a randomly selected student in a test of a probability course, be a normal random variable. A professor is said to grade such a test on the curve if he finds the average μ and the standard deviation σ of the grades and then assigns letter grades according to the following table

Range of the grade	$X \geq \mu+\sigma$	$\mu \leq X < \mu+\sigma$	$\mu-\sigma \leq X < \mu$	$\mu-2\sigma \leq X < \mu-\sigma$	$X < \mu-2\sigma$
Letter grade	A	B	C	D	F

Given that $\Phi(-2) \approx 0.0228$, $\Phi(1) \approx 0.8413$, *compute the percentage of the students who will get an F.*

Example 3

Example

Suppose that a Scottish soldiers chest size is normally distributed with mean 39.8 and standard deviation 2.05 inches, respectively. What is the probability that of 20 randomly selected Scottish soldiers, five have a chest of at least 40 inches?

Example 4

Example

The scores on an achievement test given to 100,000 students are normally distributed with mean 500 and standard deviation 100. What should the score of a student be to place him among the top 10% of all students?

$\Phi(z)$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
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1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767