Probability Theory and Simulation Methods

Lecture 22: Bivariate distributions

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	Chapters 4, 6: Random variables
Week 9	Chapter 5, 7: Special distributions
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

What you need today

- Basic computations with discrete random variables
- Independence and conditional probability
- Special discrete distributions

Joint probability mass functions

Definition *Let X and Y be two discrete random variables defined on the same sample space. Let the sets of possible values of X and Y be A and B, respectively. The function*

$$p(x, y) = P(X = x, Y = y)$$

is called the ***joint probability mass function*** *of X and Y.*

Note that $p(x, y) \geq 0$. If $x \notin A$ or $y \notin B$, then $p(x, y) = 0$. Also,

$$\sum_{x \in A} \sum_{y \in B} p(x, y) = 1. \tag{8.1}$$

Joint probability mass functions

Let X and Y have joint probability mass function $p(x, y)$. Let p_X be the probability mass function of X. Then

$$p_X(x) = P(X = x) = P(X = x, Y \in B)$$
$$= \sum_{y \in B} P(X = x, Y = y) = \sum_{y \in B} p(x, y).$$

Similarly, p_Y, the probability mass function of Y, is given by

$$p_Y(y) = \sum_{x \in A} p(x, y).$$

Marginal probability mass functions

Definition *Let X and Y have joint probability mass function $p(x, y)$. Let A be the set of possible values of X and B be the set of possible values of Y. Then the functions $p_X(x) = \sum_{y \in B} p(x, y)$ and $p_Y(y) = \sum_{x \in A} p(x, y)$ are called, respectively, the **marginal probability mass functions** of X and Y.*

Example

Example

- Roll a balanced die and let the outcome be X
- Then toss a fair coin X times and let Y denote the number of tails
- Denote the joint probability mass function of X and Y by $p(x, y)$.

Compute:

- $p(1, 0)$
- $p(1, 2)$

Solution

By definition

$$\begin{aligned} p(1,0) &= P(X = 1, Y = 0) \\ &= P(X = 1) \cdot P(Y = 0 \mid X = 1) \\ &= \frac{1}{6} \cdot \frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

Similarly

$$p(1,2) = 0$$

Joint probability mass function, presented as a table

	y						
x	0	1	2	3	4	5	6
1	1/12	1/12	0	0	0	0	0
2	1/24	2/24	1/24	0	0	0	0
3	1/48	3/48	3/48	1/48	0	0	0
4	1/96	4/96	6/96	4/96	1/96	0	0
5	1/192	5/192	10/192	10/192	5/192	1/192	0
6	1/384	6/384	15/384	20/384	15/384	6/384	1/384

Quest 1

- Your team receive a dice from Professor Willow
- Let X be the random variables that denotes the outcome of a random roll of your dice
- Then toss a fair coin X times and let Y denote the number of tails
- Denote the joint probability mass function of X and Y by $p(x, y)$

Quest: Work with your teammates to present $p(x, y)$ as a table

Joint probability mass function, presented as a table

x	y							$p_X(x)$
	0	1	2	3	4	5	6	
1	1/12	1/12	0	0	0	0	0	1/6
2	1/24	2/24	1/24	0	0	0	0	1/6
3	1/48	3/48	3/48	1/48	0	0	0	1/6
4	1/96	4/96	6/96	4/96	1/96	0	0	1/6
5	1/192	5/192	10/192	10/192	5/192	1/192	0	1/6
6	1/384	6/384	15/384	20/384	15/384	6/384	1/384	1/6
$p_Y(y)$	63/384	120/384	99/384	64/384	29/384	8/384	1/384	

Quest 2

- Construct the marginal probability mass functions of X and Y
- Compute $E(Y)$

Independence of discrete random variables

Definition

Let X and Y be two discrete random variables defined on the same sample space. If $p(x, y)$ is the joint probability mass function of X and Y, then X and Y are independent if and only if for all real numbers x and y,

$$p(x, y) = p_X(x) \cdot p_Y(y)$$

Conditional probability of discrete random variables

- Let X be a discrete random variable with set of possible values A, and let Y be a discrete random variable with set of possible values B.
- Let p(x, y) be the joint probability mass function of X and Y , and let p_X and p_Y be the marginal probability mass functions of X and Y
- If no information about the value of Y is given, the probability mass function of X is p_X

Conditional probability of discrete random variables

- If the value of Y is known, then instead of pX(x), the **conditional probability mass function of X given that** $Y = y$ is used
- This function, denoted by $p_{X|Y}(x|y)$, is defined by

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}$$

- Similarly

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{p(x, y)}{p_X(x)}$$

Quest 3

- Check if X and Y are independent
- Let a be the smaller number on your dice, compute the conditional probability mass function given that $Y = a$

$$p_{X|Y}(x, a)$$

- Compute

$$E[X|Y = a]$$

Example 2

Example

Let the joint probability mass function of X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{70}x(x+y) & \text{if } x = 1,2,3, \quad y = 3,4 \\ 0 & \text{elsewhere} \end{cases}$$

Compute $E(X)$ and $E(Y)$. Are X and Y independent?