Probability Theory and Simulation Methods

April 27th, 2018

Lecture 28: Covariance

Final exam: May 21st (1pm-3pm)

Week 1	Chapter 1: Axioms of probability
Week 2	Chapter 3: Conditional probability and independence
Week 4	Chapters 4, 6: Random variables
Week 9	Chapter 5, 7: Special distributions
Week 10	Chapters 8, 9, 10: Bivariate and multivariate distributions
Week 12	Chapter 11: Limit theorems

Chapter 9 and 10

Review: Bivariate distributions

Joint probability mass functions

Definition *Let X and Y be two discrete random variables defined on the same sample space. Let the sets of possible values of X and Y be A and B, respectively. The function*

$$p(x, y) = P(X = x, Y = y)$$

is called the ***joint probability mass function*** *of X and Y.*

Note that $p(x, y) \geq 0$. If $x \notin A$ or $y \notin B$, then $p(x, y) = 0$. Also,

$$\sum_{x \in A} \sum_{y \in B} p(x, y) = 1. \tag{8.1}$$

Marginal probability mass functions

Definition *Let X and Y have joint probability mass function $p(x, y)$. Let A be the set of possible values of X and B be the set of possible values of Y. Then the functions $p_X(x) = \sum_{y \in B} p(x, y)$ and $p_Y(y) = \sum_{x \in A} p(x, y)$ are called, respectively, the* ***marginal probability mass functions*** *of X and Y.*

Joint probability density functions

Definition *Two random variables X and Y, defined on the same sample space, have a continuous joint distribution if there exists a nonnegative function of two variables, $f(x, y)$ on $\mathbf{R} \times \mathbf{R}$, such that for any region R in the xy-plane that can be formed from rectangles by a countable number of set operations,*

$$P\big((X, Y) \in R\big) = \iint_R f(x, y)\, dx\, dy. \tag{8.2}$$

*The function $f(x, y)$ is called the **joint probability density function** of X and Y.*

Marginal probability density functions

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx. \tag{8.5}$$

Similarly,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy. \tag{8.6}$$

Therefore, it is reasonable to make the following definition:

Definition *Let X and Y have joint probability density function $f(x, y)$; then the functions f_X and f_Y, given by* (8.6) *and* (8.5), *are called, respectively, the* ***marginal probability density functions*** *of X and Y.*

The most important questions

When we read an probabilistic expression, e.g

$$E[X+2Y],\ E[X],\ Var[Y],\ E[X|Y]$$

we need to answer the following questions:

- What is the probability distribution we are working with: is that the joint distribution, is that the marginal distribution, is that the conditional distribution?
- Are the random variables continuous or discrete? Should I use summation or integration?

Example

Theorem

Let X and Y be two continuous random variables with joint probability density function $f(x, y)$. Prove that

$$E(X + Y) = E(X) + E(Y)$$

Example

Theorem

Let X and Y be two discrete random variables. Then

$$E[E(X|Y)] = E(X)$$

Example

Theorem

Let X and Y be two discrete random variables. Then

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

Expected values of sums of random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be random variables on the same sample space. Then

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots E[X_n]$$

Covariance

Definition

Let X and Y be two random variables; then the covariance of X and Y is defined by

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

Computing formula

Theorem

$$Cov(X, Y) = E[XY] - E[X] \cdot E[Y]$$

Hint: For convenience, define the constants

$$\mu_1 = E(X), \mu_2 = E(Y),$$

factor the expression of $Cov(X, Y)$, and use the linearity of expectation.

Example

Let X and Y be two random variables with the following joint distribution

p(x,y)		x = 0	x = 1
y	0	0.1	0.5
	1	0.2	0.2

Compute the covariance between X and Y

Example

Example

Let X be uniformly distributed over $(-1, 1)$ and $Y = X^2$.
Compute $Cov(X, Y)$

Recall that the uniform distribution on (a, b) has density

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a, b) \\ 0 & \text{elsewhere} \end{cases}$$

Computing formula

Problem

Prove that

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$$