MATH 450: Mathematical statistics

September 3rd, 2019

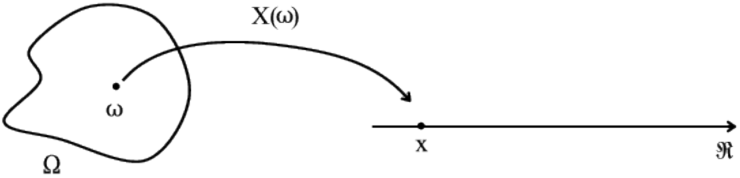
Lecture 3: Statistics and Sampling Distributions

Topics

Week 2	*Chapter 6: Statistics and Sampling Distributions*
Week 4	Chapter 7: Point Estimation
Week 6	*Chapter 8: Confidence Intervals*
Week 9	*Chapter 9: Test of Hypothesis*
Week 11	Chapter 10: Two-sample inference
Week 12	Regression

Week 1: Probability review

Random variable



Definition

Let S be the sample space of an experiment. A real-valued function $X : S \to \mathbb{R}$ is called a random variable of the experiment.

Discrete random variable

A random variable X is described by its *probability mass function*

Definition *The* ***probability mass function*** *p of a random variable X whose set of possible values is $\{x_1, x_2, x_3, \dots\}$ is a function from $\mathbf{R}$ to $\mathbf{R}$ that satisfies the following properties.*

(a) $p(x) = 0$ *if* $x \notin \{x_1, x_2, x_3, \dots\}$.

(b) $p(x_i) = P(X = x_i)$ *and hence* $p(x_i) \geq 0$ $(i = 1, 2, 3, \dots)$.

(c) $\sum_{i=1}^{\infty} p(x_i) = 1$.

Law of the unconscious statistician (LOTUS)

Theorem 4.2 *Let X be a discrete random variable with set of possible values A and probability mass function $p(x)$, and let g be a real-valued function. Then $g(X)$ is a random variable with*

$$E[g(X)] = \sum_{x \in A} g(x)p(x).$$

Continuous random variable

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **absolutely continuous** or, for simplicity, **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

Whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies the equation above.

Properties

Let X be a continuous r.v. with density function f, then

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\ dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

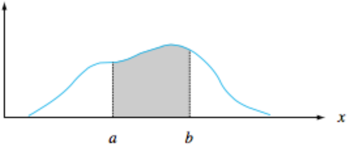


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Definition *If X is a continuous random variable with probability density function f, the* ***expected value*** *of X is defined by*

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

The expected value of X is also called the **mean**, or **mathematical expectation**, or simply the **expectation** of X, and as in the discrete case, sometimes it is denoted by EX, $E[X]$, μ, or μ_X.

Lotus

Theorem 6.3 *Let X be a continuous random variable with probability density function $f(x)$; then for any function $h: \mathbf{R} \to \mathbf{R}$,*

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx.$$

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.

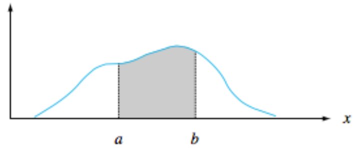


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

For continuous random variable:

$$P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

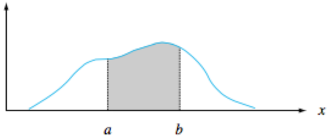
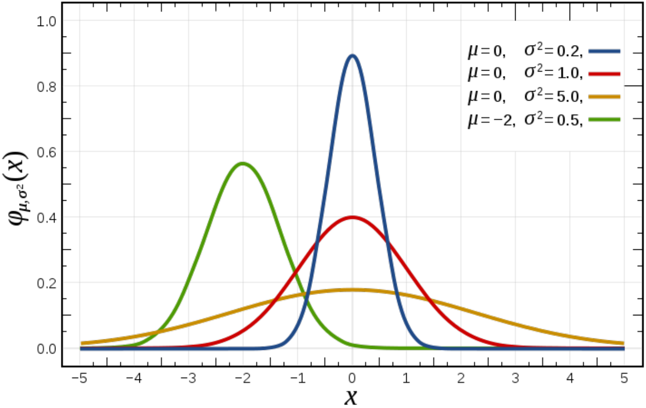


Figure 4.2 $P(a \le X \le b)$ = the area under the density curve between a and b

Moreover:

$$f(x) = F'(x)$$

$\mathcal{N}(\mu, \sigma^2)$



$$E(X) = \mu, \; Var(X) = \sigma^2$$

Standard normal distribution $\mathcal{N}(0, 1)$

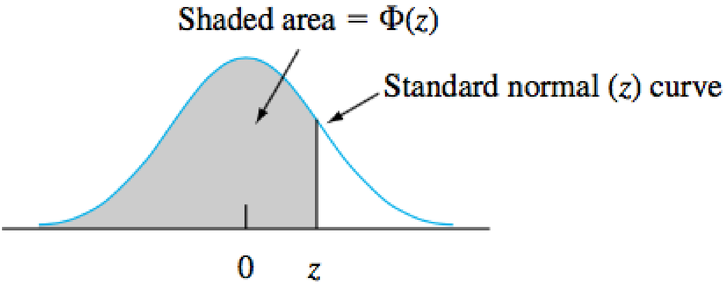
- If Z is a normal random variable with parameters $\mu = 0$ and $\sigma = 1$, then the pdf of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

  and Z is called the *standard normal distribution*

- $E(Z) = 0, \; Var(Z) = 1$

$\Phi(z)$



$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(y)\, dy$$

$\Phi(z)$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Shifting and scaling normal random variables

If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution. Thus

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X \le a) = \Phi\left(\frac{a-\mu}{\sigma}\right) \quad P(X \ge b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$$

Exercise 3

Problem

Let X be a $\mathcal{N}(3, 9)$ random variable. Compute $P[X \leq 5.25]$.

Descriptive statistics

1.3: Measures of locations

- The Mean
- The Median
- Trimmed Means

Measures of locations: mean

The **sample mean** $\bar{x}$ of observations $x_1, x_2, \ldots, x_n$ is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Measures of locations: median

Step 1: ordering the observations from smallest to largest

$$\tilde{x} = \begin{cases} \text{The single middle value if } n \text{ is odd} & = \left(\dfrac{n+1}{2}\right)^{\text{th}} \text{ordered value} \\ \text{The average of the two middle values if } n \text{ is even} & = \text{average of } \left(\dfrac{n}{2}\right)^{\text{th}} \text{ and } \left(\dfrac{n}{2}+1\right)^{\text{th}} \text{ ordered values} \end{cases}$$

Median is not affected by outliers

Measures of locations: trimmed mean

- A $\alpha\%$ trimmed mean is computed by:
  - eliminating the smallest $\alpha\%$ and the largest $\alpha\%$ of the sample
  - averaging what remains
- $\alpha = 0 \rightarrow$ the mean
- $\alpha \approx 50 \rightarrow$ the median

Measures of variability: deviations from the mean

The **sample variance**, denoted by s^2, is given by

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1} = \frac{S_{xx}}{n - 1}$$

The **sample standard deviation**, denoted by s, is the (positive) square root of the variance:

$$s = \sqrt{s^2}$$

Working with vectors in R

- manually create a vector a with entry values

$$a = c(1, 2, 6, 8, 5, 3, -1, 2.1, 0)$$

- create a zero vector with length $n = 25$

$$a = rep(0, 25)$$

- $a[i]$ is the i^{th} element of a
- manipulate all entries at the same time using 'for' loop

Working with vectors in R

- *rnorm*(n, mean=0, sd=2)
  generate a vector of n observations withdraw from the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 2$
- *hist*(A)
  produce a histogram plot of the vector A
- *boxplot*(A)
  produce a boxplot of A
  `https://www.rdocumentation.org/packages/graphics/versions/3.6.1/topics/boxplot`

Boxplots

Order the n observations from smallest to largest and separate the smallest half from the largest half; the median $\tilde{x}$ is included in both halves if n is odd. Then the **lower fourth** is the median of the smallest half and the **upper fourth** is the median of the largest half. A measure of spread that is resistant to outliers is the **fourth spread** f_s, given by

$$f_s = \text{upper fourth} - \text{lower fourth}$$

40 52 55 60 70 75 85 85 90 90 92 94 94 95 98 100 115 125 125

The five-number summary is as follows:

smallest $x_i = 40$ lower fourth $= 72.5$ $\tilde{x} = 90$ upper fourth $= 96.5$
largest $x_i = 125$

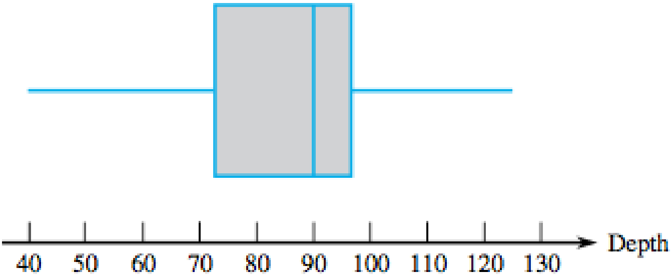
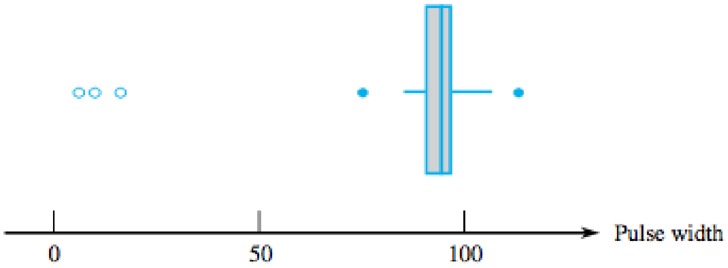


Figure 1.17 A boxplot of the corrosion data

Boxplot with outliers

Any observation farther than $1.5f_s$ from the closest fourth is an **outlier**. An outlier is **extreme** if it is more than $3f_s$ from the nearest fourth, and it is **mild** otherwise.



Statistics and sampling distribution

Overview

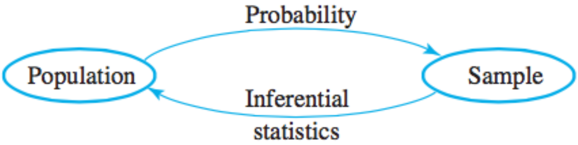
6.1 Statistics and their distributions

6.2 The distribution of the sample mean

6.3 The distribution of a linear combination

Order $6.1 \to 6.3 \to 6.2$

Random sample



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

Recap: Independent random variables

Definition

Two random variables X and Y are said to be independent if for every pair of x and y values,

$$P(X = x, Y = y) = P_X(x) \cdot P_Y(y) \quad \text{if the variables are discrete}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{if the variables are continuous}$$

Property

If X and Y are independent, then for any functions g and h

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$$

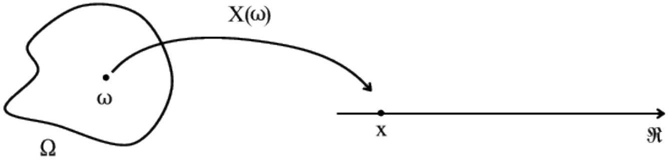
Statistics

Definition

A statistic is any quantity whose value can be calculated from sample data

- prior to obtaining data, there is uncertainty as to what value of any particular statistic will result $\rightarrow$ a statistic is a random variable
- the probability distribution of a statistic is referred to as its *sampling distribution*

Random variables



- random variables are used to model uncertainties
- Notations:
  - random variables are denoted by uppercase letters (e.g., X);
  - the calculated/observed values of the random variables are denoted by lowercase letters (e.g., x)

Example of a statistic

- Let $X_1, X_2, \ldots, X_n$ be a random sample of size n
- The sample mean of $X_1, X_2, \ldots, X_n$, defined by

$$\bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n},$$

is a statistic

- When the values of $x_1, x_2, \ldots, x_n$ are collected,

$$\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n},$$

is a realization of the statistic $\bar{X}$

Example of a statistic

- Let $X_1, X_2, \ldots, X_n$ be a random sample of size n
- The random variable

  $$T = X_1 + 2X_2 + 3X_5$$

  is a statistic
- When the values of $x_1, x_2, \ldots, x_n$ are collected,

  $$t = x_1 + 2x_2 + 3x_5,$$

  is a realization of the statistic T

Questions for this chapter

Given statistic T computed from sample $X_1, X_2, \ldots, X_n$

- Question 1: If we **know** the distribution of X_i's, can we obtain the distribution of T?
- Question 2: If we **don't know** the distribution of X_i's, can we still obtain/approximate the distribution of T?

Questions for this chapter

Real questions: If T is a linear combination of X_i's, can we

- compute the distribution of T in some easy cases?
- compute the expected value and variance of T?

Questions for this section

Real questions: If $T = X_1 + X_2$

- compute the distribution of T in some easy cases
- compute the expected value and variance of T

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P, *and* $T = X_1 + X_2$.

1. *Compute* $P[T = 40]$

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P, *and* $T = X_1 + X_2$.

1. *Compute* $P[T = 40]$
2. *Derive the probability mass function of* T

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P, *and* $T = X_1 + X_2$.

1. *Compute* $P[T = 100]$
2. *Derive the probability mass function of* T
3. *Compute the expected value and the standard deviation of* T

Example 2

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$.
What is the distribution of T*?*

cdf

For continuous random variable:

$$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\, dx$$

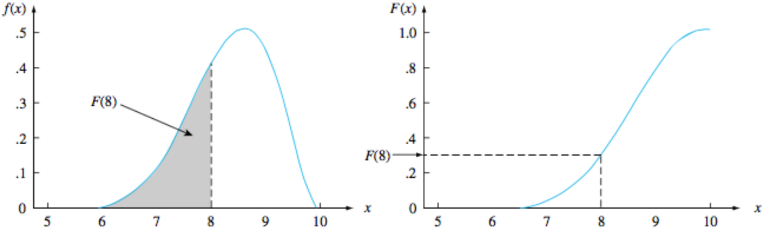


Figure 4.5 A pdf and associated cdf

Moreover:

$$f(x) = F'(x)$$

Example 2

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$.

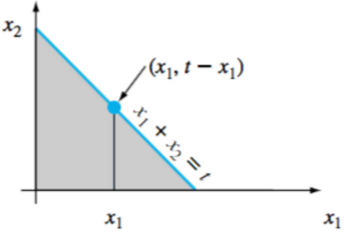
1. *Compute the cumulative density function (cdf) of* T

Example 2

$$F_{T_o}(t) = P(X_1 + X_2 \le t) = \iint_{\{(x_1, x_2): x_1 + x_2 \le t\}} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^t \int_0^{t-x_1} \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2}\, dx_2\, dx_1 = \int_0^t (\lambda e^{-\lambda x_1} - \lambda e^{-\lambda t})\, dx_1$$

$$= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$$



Example 2b

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* $\lambda = 2$

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$.

1. *Compute the cumulative density function (cdf) of* T
2. *Compute the probability density function (pdf) of* T

Summary

1. If the distribution and the statistic T is simple, try to construct the pmf of the statistic (as in Example 1)
2. If the probability density function $f_X(x)$ of X's is known, the
   - try to represent/compute the cumulative distribution (cdf) of T
   $$\mathbb{P}[T \le t]$$
   - take the derivative of the function (with respect to t)

Example 1*

Problem

Consider the distribution P

x	40	45	50
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P*, and* $T = X_1 - X_2$.

1. *Derive the probability mass function of T*
2. *Compute the expected value and the standard deviation of T*

Example 2*

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + 2X_2$.

1. *Compute the cumulative density function (cdf) of* T
2. *Compute the probability density function (pdf) of* T