MATH 450: Mathematical statistics

September 5th, 2019

Lecture 4: Statistics and their distributions

Topics

Week 2	**Chapter 6: Statistics and Sampling Distributions**
Week 4	Chapter 7: Point Estimation
Week 6	Chapter 8: Confidence Intervals
Week 9	Chapter 9: Test of Hypothesis
Week 11	Chapter 10: Two-sample inference
Week 12	Regression

Overview

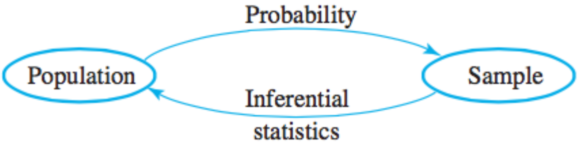
6.1 Statistics and their distributions

6.2 The distribution of the sample mean

6.3 The distribution of a linear combination

Order 6.1 $\rightarrow$ 6.3 $\rightarrow$ 6.2

Random sample



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

Recap: Independent random variables

Definition

Two random variables X and Y are said to be independent if for every pair of x and y values,

$$P(X = x, Y = y) = P_X(x) \cdot P_Y(y) \quad \text{if the variables are discrete}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{if the variables are continuous}$$

Property

If X and Y are independent, then for any functions g and h

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$$

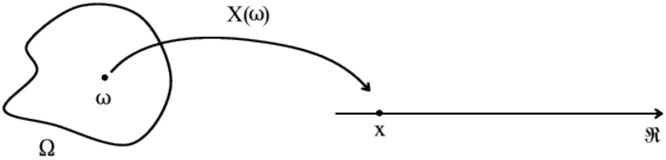
Statistics

Definition

A statistic is any quantity whose value can be calculated from sample data

- prior to obtaining data, there is uncertainty as to what value of any particular statistic will result $\rightarrow$ a statistic is a random variable
- the probability distribution of a statistic is referred to as its *sampling distribution*

Random variables



- random variables are used to model uncertainties
- Notations:
  - random variables are denoted by uppercase letters (e.g., X);
  - the calculated/observed values of the random variables are denoted by lowercase letters (e.g., x)

Example of a statistic

- Let $X_1, X_2, \ldots, X_n$ be a random sample of size n
- The sample mean of $X_1, X_2, \ldots, X_n$, defined by

$$\bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n},$$

is a statistic

- When the values of $x_1, x_2, \ldots, x_n$ are collected,

$$\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n},$$

is a realization of the statistic $\bar{X}$

Example of a statistic

- Let $X_1, X_2, \ldots, X_n$ be a random sample of size n
- The random variable

$$T = X_1 + 2X_2 + 3X_5$$

  is a statistic
- When the values of $x_1, x_2, \ldots, x_n$ are collected,

$$t = x_1 + 2x_2 + 3x_5,$$

  is a realization of the statistic T

Questions for this chapter

Given statistic T computed from sample $X_1, X_2, \ldots, X_n$

- Question 1: If we **know** the distribution of X_i's, can we obtain the distribution of T?
- Question 2: If we **don't know** the distribution of X_i's, can we still obtain/approximate the distribution of T?

Questions for this chapter

Real questions: If T is a linear combination of X_i's, can we

- compute the distribution of T in some easy cases?
- compute the expected value and variance of T?

Questions for this section

Real questions: If $T = X_1 + X_2$

- compute the distribution of T in some easy cases
- compute the expected value and variance of T

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P, *and* $T = X_1 + X_2$.

1. *Compute* $P[T = 40]$

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P*, and* $T = X_1 + X_2$.

1. *Compute* $P[T = 40]$
2. *Derive the probability mass function of* T

Example 1

Problem

Consider the distribution P

x	10	15	20
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P*, and* $T = X_1 + X_2$.

1. *Compute* $P[T = 100]$
2. *Derive the probability mass function of* T
3. *Compute the expected value and the standard deviation of* T

Example 2

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$*.*
What is the distribution of T*?*

For continuous random variable:

$$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\,dx$$

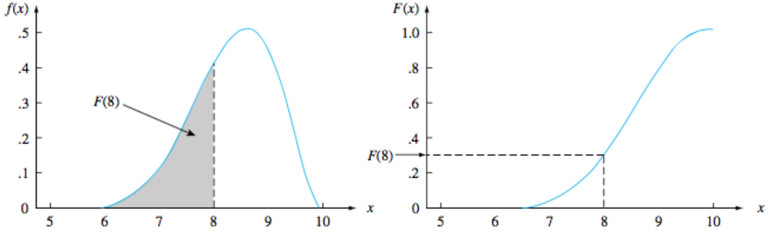


Figure 4.5 A pdf and associated cdf

Moreover:

$$f(x) = F'(x)$$

Example 2

Problem

Let $\{X_1, X_2\}$ be a random sample of size 2 from the exponential distribution with parameter λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$.

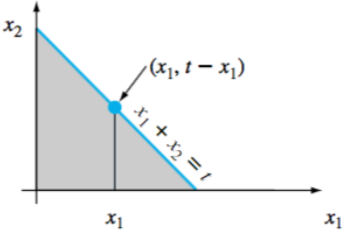
1. *Compute the cumulative density function (cdf) of T*

Example 2

$$F_{T_o}(t) = P(X_1 + X_2 \le t) = \iint\limits_{\{(x_1, x_2): x_1 + x_2 \le t\}} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^t \int_0^{t-x_1} \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2}\, dx_2\, dx_1 = \int_0^t (\lambda e^{-\lambda x_1} - \lambda e^{-\lambda t})\, dx_1$$

$$= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$$



Example 2b

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* $\lambda = 2$

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + X_2$.

1. *Compute the cumulative density function (cdf) of* T
2. *Compute the probability density function (pdf) of* T

Summary

1. If the distribution and the statistic T is simple, try to construct the pmf of the statistic (as in Example 1)
2. If the probability density function $f_X(x)$ of X's is known, the
   - try to represent/compute the cumulative distribution (cdf) of T
   $$\mathbb{P}[T \leq t]$$
   - take the derivative of the function (with respect to t)

Example 1*

Problem

Consider the distribution P

x	40	45	50
$p(x)$	0.2	0.3	0.5

Let $\{X_1, X_2\}$ *be a random sample of size 2 from* P*, and* $T = X_1 - X_2$.

1. *Derive the probability mass function of T*
2. *Compute the expected value and the standard deviation of T*

Example 2*

Problem

Let $\{X_1, X_2\}$ *be a random sample of size 2 from the exponential distribution with parameter* λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and $T = X_1 + 2X_2$.

1. *Compute the cumulative density function (cdf) of* T
2. *Compute the probability density function (pdf) of* T

Linear combination of normal random variables

Linear combination of normal random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent normal random variables (with possibly different means and/or variances). Then

$$T = a_1X_1 + a_2X_2 + \ldots a_nX_n$$

also follows the normal distribution.

What are the mean and the standard deviation of T?

- $E(T) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$
- $\sigma_T^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + \ldots + a_n^2\sigma_{X_n}^2$

Moment generating function

Moment generating function

Definition

The moment generating function (mgf) of a continuous random variable X is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

Reading: 3.4 and 4.2

Moment generating function

Property

Two distributions have the same pdf if and only if they have the same moment generating function

Moment generating function

Distribution	Moment-generating function $M_X(t)$
Bernoulli $P(X=1)=p$	$1-p+pe^t$
Geometric $(1-p)^{k-1}\,p$	$\dfrac{pe^t}{1-(1-p)e^t}$ $\forall t < -\ln(1-p)$
Binomial B(n, p)	$\left(1-p+pe^t\right)^n$
Poisson Pois(λ)	$e^{\lambda(e^t-1)}$
Uniform (continuous) U(a, b)	$\dfrac{e^{tb}-e^{ta}}{t(b-a)}$
Uniform (discrete) U(a, b)	$\dfrac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$
Normal $N(\mu, \sigma^2)$	$e^{t\mu+\frac{1}{2}\sigma^2t^2}$
Chi-squared χ^2_k	$(1-2t)^{-\frac{k}{2}}$
Gamma $\Gamma(k, \theta)$	$(1-t\theta)^{-k}; \forall t < \dfrac{1}{\theta}$
Exponential Exp(λ)	$\left(1-t\lambda^{-1}\right)^{-1}, \ (t<\lambda)$

Moment generating function

Definition

Let X_1, X_2 be a 2 independent random variables and $T = X_1 + X_2$, then

$$M_T(t) = M_{X_1}(t) M_{X_2}(t)$$

Hint:

$$M_T(t) = E(e^{tT}) = E(e^{t(X_1+X_2)}) = E(e^{tX_1} \cdot e^{tX_2})$$

Example 3

Problem

Given that the mgf of a Poisson variables with mean λ is

$$e^{\lambda(e^t-1)}$$

Suppose X and Y are independent Poisson random variables, where X has mean a and Y has mean b. Show that $T = X + Y$ also follows the Poisson distribution.

Example 4

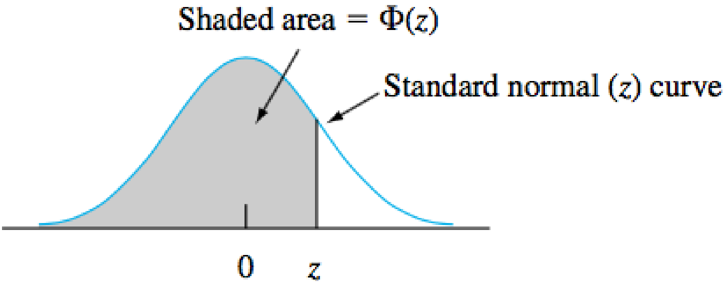
Problem

Given that the mgf of a normal random variables with mean μ and variance σ^2 is

$$e^{\mu t + \frac{\sigma^2}{2} t^2}$$

Suppose X and Y are independent normal random variables. Show that $T = X + Y$ also follows the normal distribution.

$\Phi(z)$



$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(y)\, dy$$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \le z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Example 1

Problem

Let $X_1, X_2, \ldots, X_{16}$ be a random sample from $\mathcal{N}(1, 4)$ (that is, normal distribution with mean $\mu = 1$ and standard deviation $\sigma = 2$).
Let $\bar{X}$ be the sample mean

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_{16}}{16}$$

- *What is the distribution of $\bar{X}$?*
- *Compute $P[\bar{X} \leq 1.82]$*

Example 1*

Problem

Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$ (that is, normal distribution with mean μ and standard deviation σ).
Let $\bar{X}$ be the sample mean

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

What is the distribution of $\bar{X}$?