MATH 450: Mathematical statistics

September 17th, 2019

Lecture 7: Introduction to parameter estimation

Topics

Week 2	*Chapter 6: Statistics and Sampling Distributions*
Week 4	Chapter 7: Point Estimation
Week 6	*Chapter 8: Confidence Intervals*
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Week 11	Chapter 10: Two-sample inference
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The Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then, in the limit when $n \to \infty$, the standardized version of $\bar{X}$ have the standard normal distribution

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z\right) = \mathbb{P}[Z \leq z] = \Phi(z)$$

Rule of Thumb:

If $n > 30$, the Central Limit Theorem can be used for computation.

Example

Problem

When a batch of a certain chemical product is prepared, the amount of a particular impurity in the batch is a random variable with mean value 4.0 g and standard deviation 1.5 g.

If 50 batches are independently prepared, what is the (approximate) probability that the sample average amount of impurity X is between 3.5 and 3.8 g?

Hint:

- First, compute $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}$
- Note that
$$\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}$$
is (approximately) standard normal.

Example

Problem

A restaurant reports that the tip percentage at their restaurant has a mean value of 18% and a standard deviation of 6%.

What is the approximate probability that the sample mean tip percentage for a random sample of 40 bills is between 16% and 20%?

Chapter 7: Overview

7.1 Point estimate

- unbiased estimator
- mean squared error

7.2 Methods of point estimation

- method of moments
- method of maximum likelihood.

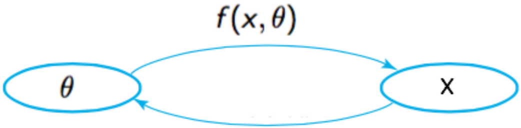
7.3 Sufficient statistic

7.4 Information and Efficiency

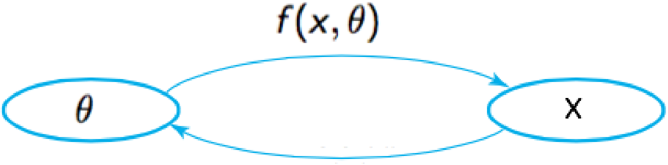
- Large sample properties of the maximum likelihood estimator

Question of this chapter

- Given a random sample $X_1, \dots, X_n$ from a distribution with pmf/pdf $f(x, \theta)$ parameterized by a parameter θ
- Goal: Estimate θ



Point estimate



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimate} \\ \theta & \Longrightarrow & X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Example

Problem

Consider a random sample $X_1, \dots, X_{10}$ from the pdf

$$f(x) = \frac{1+\theta x}{2} \qquad -1 \le x \le 1$$

Assume that the obtained data are

$$0.92, \quad -0.1, \quad -0.2, \quad 0.75, \quad 0.65, \quad -0.53$$

$$0.36, \quad -0.68, \quad 0.97, \quad -0.33, \quad 0.79$$

Provide an estimate of θ.

Mean Squared Error

- Measuring error of estimation

$$|\hat{\theta} - \theta| \quad \text{or} \quad (\hat{\theta} - \theta)^2$$

- The error of estimation is random

Definition

The mean squared error of an estimator $\hat{\theta}$ is

$$E[(\hat{\theta} - \theta)^2]$$

Problem

Let Y be a random variable and a is a constant. Prove that

$$E[(Y-a)^2] = Var(Y) + (E[Y]-a)^2$$

Hint: Recall that

$$Var[Y] = E[Y^2] - (E[Y])^2$$

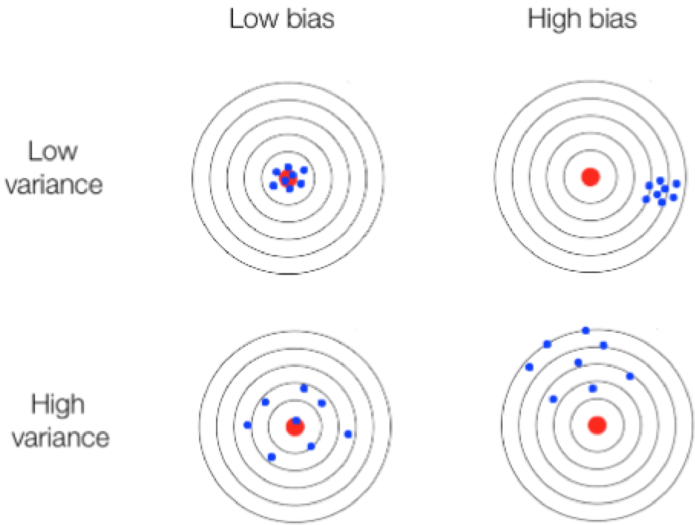
Theorem

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + \left(E(\hat{\theta}) - \theta\right)^2$$

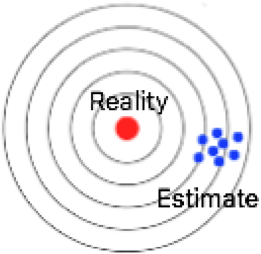
Bias-variance decomposition

$$\text{Mean squared error} = \text{variance of estimator} + (bias)^2$$

Bias-variance decomposition



Statistical bias vs. social bias



Unbiased estimators

Definition

A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

for every possible value of θ.

Unbiased estimator

$\Leftrightarrow$ Bias $= 0$

$\Leftrightarrow$ Mean squared error $=$ variance of estimator

Sample proportion

- A test is done with probability of success p. Denote the outcome by let X (success: 1, failure: 0)

$$E[X] = p, \quad Var[X] = p(1-p)$$

- n independent tests are done, let $X_1, X_2, \ldots, X_n$ be the outcomes
- Let

$$\hat{p} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

- We know that

$$E[\hat{p}] = p$$

thus $\hat{p}$ is an unbiased estimator

- Compute $MSE(\hat{p})$