MATH 450: Mathematical statistics

September 24th, 2019

Lecture 9: Method of moments (cont.)

Topics

Week 2	*Chapter 6: Statistics and Sampling Distributions*
Week 4	Chapter 7: Point Estimation
Week 7	*Chapter 8: Confidence Intervals*
Week 10	*Chapter 9: Test of Hypothesis*
Week 11	Chapter 10: Two-sample inference
Week 13	Regression

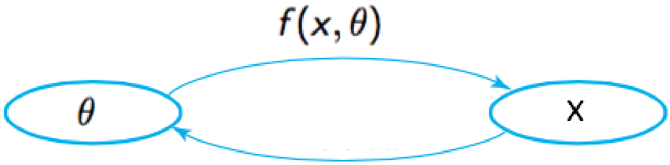
7.1 Point estimate
- unbiased estimator
- mean squared error

7.2 Methods of point estimation
- method of moments
- method of maximum likelihood.

7.3 Sufficient statistic

7.4 Information and Efficiency
- Large sample properties of the maximum likelihood estimator

Point estimate



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimate} \\ \theta & \Longrightarrow & X_1, X_2, \dots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Mean Squared Error

- Measuring error of estimation

$$|\hat{\theta} - \theta| \qquad \text{or} \qquad (\hat{\theta} - \theta)^2$$

- The error of estimation is random

Definition

The mean squared error of an estimator $\hat{\theta}$ is

$$E[(\hat{\theta} - \theta)^2]$$

Theorem

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + \left(E(\hat{\theta}) - \theta\right)^2$$

Bias-variance decomposition

Mean squared error = variance of estimator + $(bias)^2$

Unbiased estimators

Definition

A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

for every possible value of θ.

Unbiased estimator

$\Leftrightarrow$ Bias $= 0$

$\Leftrightarrow$ Mean squared error $=$ variance of estimator

Minimum variance unbiased estimator (MVUE)

Definition

Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting $\hat{\theta}$ is called the minimum variance unbiased estimator (MVUE) of θ.

Recall:

- Mean squared error $=$ variance of estimator $+$ $(\textit{bias})^2$
- unbiased estimator $\Rightarrow$ bias $=0$

$\Rightarrow$ MVUE has minimum mean squared error among unbiased estimators

What is the best estimator of the mean?

Question: Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ. What is the best estimator of μ?

Answer: It depends.

- Normal distribution $\rightarrow$ sample mean $\bar{X}$
- Cauchy distribution $\rightarrow$ sample median $\tilde{X}$
- Uniform distribution $\rightarrow$

$$\hat{X}_e = \frac{\text{largest number} + \text{smaller number}}{2}$$

- In all cases, 10% trimmed mean performs pretty well

MVUE of normal distributions

Theorem

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ. Then the estimator $\hat{\mu} = \bar{X}$ is the MVUE for μ.

Method of moments

Example 1

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

A random sample of ten students yields data

$$x_1 = .92,\ x_2 = .79,\ x_3 = .90,\ x_4 = .65,\ x_5 = .86,$$

$$x_6 = .47,\ x_7 = .73,\ x_8 = .97,\ x_9 = .94,\ x_{10} = .77$$

Provide an estimator of θ.

Idea

- We can compute $E(X) \to$ the answer will be a function of θ
- For large n, we have

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

  is close to $E[X]$
- We can compute $\bar{x}$ from the data $\to$ approximate λ

Example 1: Step 1

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute $E[X]$.

Example 1

Problem

A random sample of ten students yields data

$$x_1 = .92,\ x_2 = .79,\ x_3 = .90,\ x_4 = .65,\ x_5 = .86,$$

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Compute $\bar{x}$.

Example 1

Problem

Let $X_1, \ldots, X_{10}$ *be a random sample from a distribution with pdf*

$$f(x) = \begin{cases} (\theta+1)x^\theta & \textit{if } 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

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Provide an estimator of θ *by the method of moments.*

Method of moments: Example 2

Problem

Suppose that for a parameter $0 \leq \theta \leq 1$, X is the outcome of the roll of a four-sided tetrahedral die

x	1	2	3	4
$p(x)$	$\frac{3\theta}{4}$	$\frac{\theta}{4}$	$\frac{3(1-\theta)}{4}$	$\frac{(1-\theta)}{4}$

Suppose the die is rolled 10 times with outcomes

$$4, 1, 2, 3, 1, 2, 3, 4, 2, 3$$

Use the method of moments to obtain an estimator of θ.

Moments

- Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots$, the k^{th} population moment, or k^{th} moment of the distribution $f(x)$, is

$$E(X^k)$$

  - First moment: the mean
  - Second moment: $E(X^2)$

Sample moments

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots,$ the k^{th} sample moment is

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n}$$

The law of large numbers provides that when $n \to \infty$

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n} \to E(X^k)$$

Method of moments: ideas

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf
$$f(x; \theta_1, \theta_2, \ldots, \theta_m)$$
- Assume that for $k = 1, \ldots, m$
$$\hat{u}_k = \frac{X_1^k + X_2^k + \ldots + X_n^k}{n} = E(X^k)$$
- Solve the system of equations for $\theta_1, \theta_2, \ldots, \theta_m$

Method of moments: Example 3

Problem

Let $\beta > 1$ and $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the method of moments to obtain an estimator of β.

Method of moments: Example 4

Problem

Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $\mathcal{N}(0, \sigma^2)$.
Use the method of moments to obtain an estimator of σ.