MATH 450: Mathematical statistics

Oct 8th, 2019

Lecture 13: Confidence intervals

Countdown to midterm: 16 days

Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	Chapter 7: Point Estimation
Week 7	**Chapter 8: Confidence Intervals**
Week 10	Chapter 9: Test of Hypothesis
Week 11	Chapter 10: Two-sample inference
Week 13	Regression

7.1 Point estimate
- unbiased estimator
- mean squared error

7.2 Methods of point estimation
- method of moments
- method of maximum likelihood.

7.3 Sufficient statistic

7.4 Information and Efficiency

Information

Fisher information

Definition

The Fisher information $I(\theta)$ in a single observation from a pmf or pdf $f(x;\theta)$ is the variance of the random variable $U = \frac{\partial \log f(X,\theta)}{\partial \theta}$, which is

$$I(\theta) = Var\left[\frac{\partial \log f(X,\theta)}{\partial \theta}\right]$$

Note: We always have $E[U] = 0$

Example

Problem

Let X be distributed by

x	*0*	*1*
$f(x,\theta)$	$1-\theta$	θ

Compute $I(X,\theta)$.

Hint:

- If $x = 1$, then $f(x,\theta) = \theta$. Thus

$$u(x) = \frac{\partial \log f(x,\theta)}{\partial \theta} = \frac{1}{\theta}$$

- How about $x = 0$?

Example

Problem

Let X be distributed by

x	0	1
$f(x,\theta)$	$1-\theta$	θ

Compute $I(X,\theta)$.

We have

$$\begin{aligned} Var[U] &= E[U^2] - (E[U])^2 = E[U^2] \\ &= \sum_{x=0,1} U^2(x) f(x,\theta) \\ &= \frac{1}{(1-\theta)^2} \cdot (1-\theta) + \frac{1}{\theta^2} \cdot \theta \end{aligned}$$

The Cramer-Rao Inequality

Theorem

Assume a random sample $X_1, X_2, ..., X_n$ from the distribution with pmf or pdf $f(x, \theta)$ such that the set of possible values does not depend on θ. If the statistic $T = t(X_1, X_2, ..., X_n)$ is an unbiased estimator for the parameter θ, then

$$Var(T) \geq \frac{1}{n \cdot I(\theta)}$$

Proof for $n = 1$

Recall that $E[U] = 0$ and $E[T] = \theta$ (since T is an unbiased estimator of θ) we have

$$\begin{aligned} Cov(T, U) &= E[TU] - E[U] \cdot E[T] \\ &= \sum_x t(x) \frac{\partial \log f(x, \theta)}{\partial \theta} f(x, \theta) \\ &= \sum_x t(x) \frac{\partial f(x, \theta)}{\partial \theta} \frac{1}{f(x, \theta)} f(x, \theta) \\ &= \frac{\partial}{\partial \theta} \left(\sum_x t(x) f(x, \theta) \right) = 1 \end{aligned}$$

Proof for $n = 1$

The Cauchy–Schwarz inequality shows that

$$Cov(T, U) \leq \sqrt{Var(T) \cdot Var(U)}$$

which implies

$$Var(T) \geq \frac{1}{I(\theta)}.$$

Heisenberg's Uncertainty Principle

uncertainty in momentum

$$\Delta x \Delta p \geq \frac{h}{4\pi} = \frac{\hbar}{2}$$

uncertainty in position

The more accurately you know the position (i.e., the smaller Δx is), the less accurately you know the momentum (i.e., the larger Δp is); and vice versa

Efficiency

Theorem

Let $T = t(X_1, X_2, ..., X_n)$ is an unbiased estimator for the parameter θ, the ratio of the lower bound to the variance of T is its efficiency

$$\textit{Efficiency} = \frac{1}{nI(\theta)V(T)} \leq 1$$

T is said to be an efficient estimator if T achieves the Cramer–Rao lower bound (i.e., the efficiency is 1).

Note: An efficient estimator is a minimum variance unbiased (MVUE) estimator.

Large Sample Properties of the MLE

Theorem

Given a random sample $X_1, X_2, ..., X_n$ from the distribution with pmf or pdf $f(x, \theta)$ such that the set of possible values does not depend on θ. Then for large n the maximum likelihood estimator $\hat{\theta}$ has approximately a normal distribution with mean θ and variance $\frac{1}{n \cdot I(\theta)}$.
More precisely, the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is normal with mean 0 and variance $1/I(\theta)$.

Chapter 8: Confidence intervals

Overview

8.1 Basic properties of confidence intervals (CIs)
- Interpreting CIs
- General principles to derive CI

8.2 Large-sample confidence intervals for a population mean
- Using the Central Limit Theorem to derive CIs

8.3 Intervals based on normal distribution
- Using Student's t-distribution

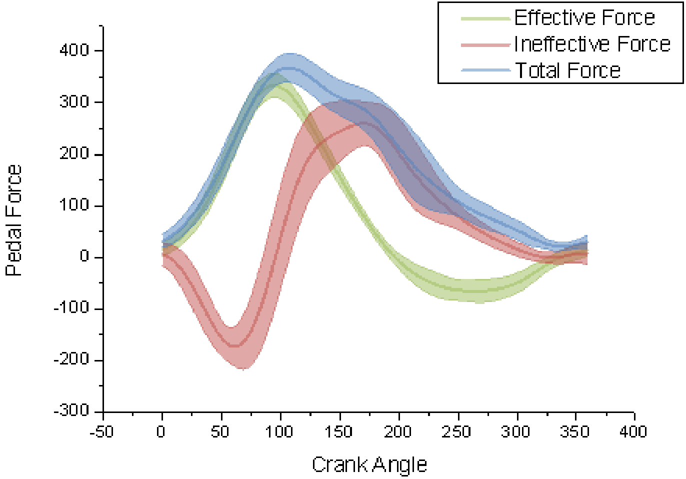
8.4 CIs for standard deviation

Framework

- Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution $f(x, \theta)$
- In Chapter 7, we learnt methods to construct an estimate $\hat{\theta}$ of θ
- Goal: we want to indicate the degree of uncertainty associated with this random prediction
- One way to do so is to construct a *confidence interval* $[\hat{\theta} - a, \hat{\theta} + b]$ such that

$$P[\theta \in [\hat{\theta} - a, \hat{\theta} + b]] = 95\%$$

Confidence interval



Principles for deriving CIs

If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution $f(x, \theta)$, then

- Find a random variable $Y = h(X_1, X_2, \ldots, X_n; \theta)$ such that the probability distribution of Y does not depend on θ or on any other unknown parameters.
- Find constants a, b such that

$$P\left[a < h(X_1, X_2, \ldots, X_n; \theta) < b\right] = 0.95$$

- Manipulate these inequalities to isolate θ

$$P\left[\ell(X_1, X_2, \ldots, X_n) < \theta < u(X_1, X_2, \ldots, X_n)\right] = 0.95$$

Confidence interval: example

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ *and standard deviation 0.85.*
If a random sample of 25 specimens is selected, with sample average $\bar{X}$.

- *Find a number a such that*

$$P[-a < \bar{X} - \mu < a] = 0.95$$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Confidence interval: example

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ and standard deviation 0.85.

- *If a random sample of 25 specimens is selected, with sample average $\bar{X}$. Find a such that*

$$P[-a < \bar{X} - \mu < a] = 0.95$$

If $\bar{x} = 2.65$, then we know with confidence 95% that

$$\mu \in (2.65 - a,\ 2.65 + a)$$

$\rightarrow$ This is a confidence interval for the population mean μ

One-sided confidence interval

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ and standard deviation 0.85.

If a random sample of 25 specimens is selected, with sample average $\bar{X}$. Find a number b such that

$$P[\bar{X} < b] = 0.95$$

8.1: Normal distribution with know σ

- Assumptions:
  - Normal distribution
  - σ is known
- 95% confidence interval
  If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

NOTATION

z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

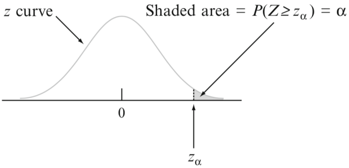


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1-\alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as **z critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

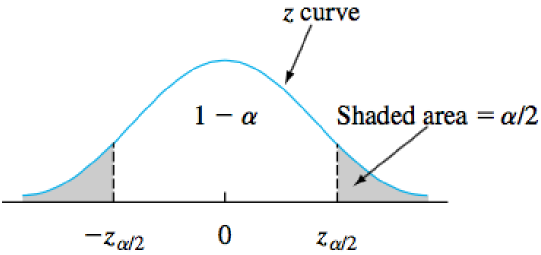


Figure 8.4 $P(-z_{\alpha/2} \le Z \le z_{\alpha/2}) = 1-\alpha$

$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

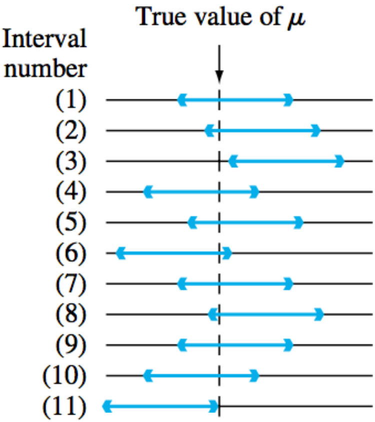
or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Assumptions

- Section 8.1
  - Normal distribution
  - σ is known
- Section 8.2
  - ~~Normal distribution~~
    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$
- Section 8.3
  - Normal distribution
  - ~~σ is known~~

  $\rightarrow$ Introducing t-distribution

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Interpreting confidence intervals

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT okay.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is okay.
- Saying "if we repeat the experiment many times, the interval contains μ about 95% of the time" is perfect.