MATH 450: Mathematical statistics

September 29th, 2020

Lecture 9: Method of maximum likelihood

Topics

Week 2	*Chapter 6: Statistics and Sampling Distributions*
Week 4	Chapter 7: Point Estimation
Week 7	*Chapter 8: Confidence Intervals*
Week 10	*Chapter 9: Test of Hypothesis*
Week 11	Chapter 10: Two-sample inference
Week 13	Regression

Chapter 7: Overview

7.1 Point estimate

- unbiased estimator
- mean squared error

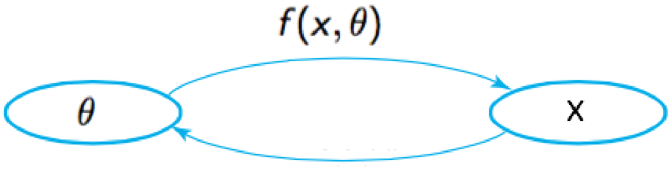
7.2 Methods of point estimation

- method of moments
- method of maximum likelihood.

7.3 Sufficient statistic

7.4 Information and Efficiency

- Large sample properties of the maximum likelihood estimator



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimator} \\ \theta & \Longrightarrow & X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Estimate vs estimator

$$\begin{array}{ccc} \textit{sample} & \Longrightarrow & \textit{estimator} \\ X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

$$\begin{array}{ccc} \textit{observed data} & \Longrightarrow & \textit{estimate} \\ x_1, x_2, \ldots, x_n & \Longrightarrow & \hat{\theta} \end{array}$$

Mean Squared Error

- Measuring error of estimation

$$|\hat{\theta} - \theta| \quad \text{or} \quad (\hat{\theta} - \theta)^2$$

- The error of estimation is random

Definition

The mean squared error of an estimator $\hat{\theta}$ is

$$E[(\hat{\theta} - \theta)^2]$$

Bias-variance decomposition

Theorem

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + \left(E(\hat{\theta}) - \theta\right)^2$$

Bias-variance decomposition

$$\text{Mean squared error} = \text{variance of estimator} + (bias)^2$$

Unbiased estimators

Definition

A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

for every possible value of θ.

Unbiased estimator

$\Leftrightarrow$ Bias $= 0$

$\Leftrightarrow$ Mean squared error $=$ variance of estimator

Method of moments

Idea

- We can compute $E(X) \to$ the answer will be a function of θ
- For large n, we have

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

  is close to $E[X]$
- We can compute $\bar{x}$ from the data $\to$ approximate λ

Example 1

Problem

Let $X_1, \ldots, X_{10}$ *be a random sample from a distribution with pdf*

$$f(x) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

A random sample of ten students yields data

$$x_1 = .92,\ x_2 = .79,\ x_3 = .90,\ x_4 = .65,\ x_5 = .86,$$

$$x_6 = .47,\ x_7 = .73,\ x_8 = .97,\ x_9 = .94,\ x_{10} = .77$$

Provide an estimate of θ.

Method of moments: Example 2

Problem

Suppose that for a parameter $0 \le \theta \le 1$, X is the outcome of the roll of a four-sided tetrahedral die

x	1	2	3	4
$p(x)$	$\frac{3\theta}{4}$	$\frac{\theta}{4}$	$\frac{3(1-\theta)}{4}$	$\frac{(1-\theta)}{4}$

Suppose the die is rolled 10 times with outcomes

$$4, 1, 2, 3, 1, 2, 3, 4, 2, 3$$

Use the method of moments to obtain an estimate of θ.

Moments

- Let $X_1, \dots, X_n$ be a random sample from a normal distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \dots$, the k^{th} population moment, or k^{th} moment of the distribution $f(x)$, is

$$E(X^k)$$

  - First moment: the mean
  - Second moment: $E(X^2)$

Sample moments

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots,$ the k^{th} sample moment is

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n}$$

The law of large numbers provides that when $n \to \infty$

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n} \to E(X^k)$$

Method of moments: ideas

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf
$$f(x; \theta_1, \theta_2, \ldots, \theta_m)$$
- Assume that for $k = 1, \ldots, m$
$$\hat{u}_k = \frac{X_1^k + X_2^k + \ldots + X_n^k}{n} = E(X^k)$$
- Solve the system of equations for $\theta_1, \theta_2, \ldots, \theta_m$

Method of moments: Example 3

Problem

Let $\beta > 1$ and $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

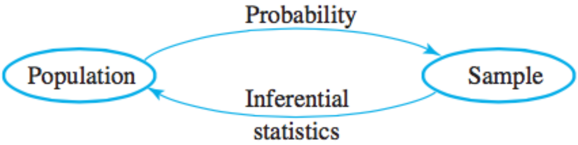
Use the method of moments to obtain an estimator of β.

Method of moments: Example 4

Problem

Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $\mathcal{N}(\mu, \sigma^2)$.
Use the method of moments to obtain estimators of μ and σ.

Method of maximum likelihood



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

Independent random variables

Definition

Two random variables X and Y are said to be independent if for every pair of x and y values,

$$P(X = x, Y = y) = P_X(x) \cdot P_Y(y) \qquad \text{if the variables are discrete}$$

or

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad \text{if the variables are continuous}$$

Random sample

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with density function $f_X(x)$.

Then the density of the joint distribution of $(X_1, X_2, ..., X_n)$ is

$$f_{joint}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

Maximum likelihood estimator

- Let $X_1, X_2, ..., X_n$ have joint pmf or pdf

$$f_{joint}(x_1, x_2, \ldots, x_n; \theta)$$

  where θ is unknown.

- When $x_1, \ldots, x_n$ are the observed sample values and this expression is regarded as a function of θ, it is called the **likelihood function**.
- The maximum likelihood estimates θ_{ML} are the value for θ that **maximize the likelihood function**:

$$f_{joint}(x_1, x_2, \ldots, x_n; \theta_{ML}) \geq f_{joint}(x_1, x_2, \ldots, x_n; \theta) \qquad \forall \theta$$

How to find the MLE?

- Step 1: Write down the likelihood function.
- Step 2: Can you find the maximum of this function?
- Step 3: Try taking the logarithm of this function.
- Step 4: Find the maximum of this new function.

To find the maximum of a function of θ:

- compute the derivative of the function with respect to θ
- set this expression of the derivative to 0
- solve the equation

Example 1

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from the exponential distribution with parameter λ, that is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

The observed data are

$$3.11, 0.64, 2.55, 2.20, 5.44,$$

$$3.42, 1.39, 8.13, 1.82, 1.30$$

Use the method of maximum likelihood to obtain an estimate of λ.

Example 2

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from a distribution with pdf

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Use the method of maximum likelihood to obtain an estimate of θ.

Example 3

Problem

Let $\beta > 1$ and $X_1, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \textit{if } x > 1 \\ 0 & \textit{otherwise} \end{cases}$$

Use the method of maximum likelihood to obtain an estimator of β.

Example 4

Problem

Let $X_1, \ldots, X_n$ *be a random sample from the normal distribution* $\mathcal{N}(0, \sigma^2)$, *that is*

$$f(x, \theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Use the method of maximum likelihood to obtain an estimator of σ.