MATH 450: Mathematical statistics

Oct 22nd, 2020

Lecture 16: Confidence intervals for standard deviation

Countdown to midterm: 7 days

Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	Chapter 7: Point Estimation
Week 7	**Chapter 8: Confidence Intervals**
Week 10	Chapter 9: Test of Hypothesis
Week 11	Chapter 10: Two-sample inference
Week 13	Regression

Overview

8.1 Basic properties of confidence intervals (CIs)
- Interpreting CIs
- General principles to derive CI

8.2 Large-sample confidence intervals for a population mean
- Using the Central Limit Theorem to derive CIs

8.3 Intervals based on normal distribution
- Using Student's t-distribution

8.4 CIs for standard deviation

Confidence intervals for a population mean

- Section 8.1: Normal distribution with known σ
  - Normal distribution
  - σ is known
- Section 8.2: Large-sample confidence intervals
  - ~~Normal distribution~~
    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$
- Section 8.3: Intervals based on normal distributions
  - Normal distribution
  - ~~σ is known~~

$\rightarrow$ Introducing t-distribution

8.1: Normal distribution with known σ

8.1: Normal distribution with known σ

- Assumptions:
  - Normal distribution
  - σ is known
- 95% confidence interval
  If after observing $X_1 = x_1$, $X_2 = x_2, \dots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

  is a 95% confidence interval of μ

z-critical value

NOTATION z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

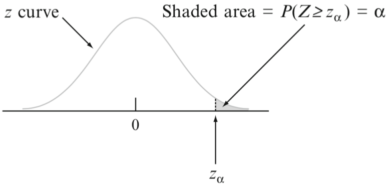


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1-\alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as ***z* critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

8.2: Large-sample CIs of the population mean

Principles

- Central Limit Theorem
$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$
is approximately normal when $n > 30$
- Moreover, when n is sufficiently large $s \approx \sigma$
- Conclusion:
$$\frac{\bar{X}-\mu}{s/\sqrt{n}}$$
is approximately normal when n is sufficiently large

If $n > 40$, we can ignore the normal assumption and replace σ by s

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

One-sided CIs

A **large-sample upper confidence bound for μ** is

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

and a **large-sample lower confidence bound for μ** is

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}$$

8.3: Intervals based on normal distributions

Assumptions

- the population of interest is normal (i.e., $X_1, \ldots, X_n$ constitutes a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$).
- σ is unknown

$\rightarrow$ we want to consider cases when n is small.

t distributions

When $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ, the rv

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has the t distribution with $n - 1$ degree of freedom (df).

Confidence intervals

Let $\bar{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal population with mean μ. Then a **100(1 − α)% confidence interval for μ**, the **one-sample t CI**, is

$$\left(\bar{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}\right) \tag{8.15}$$

or, more compactly, $\bar{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$.

An **upper confidence bound for μ** is

$$\bar{x} + t_{\alpha,n-1} \cdot \frac{s}{\sqrt{n}}$$

and replacing + by − in this latter expression gives a **lower confidence bound for μ**; both have confidence level $100(1-\alpha)\%$.

t distributions

Let $t_{\alpha,\nu}$ = the number on the measurement axis for which the area under the t curve with ν df to the right of $t_{\alpha,\nu}$, is α; $t_{\alpha,\nu}$ is called a ***t* critical value.**

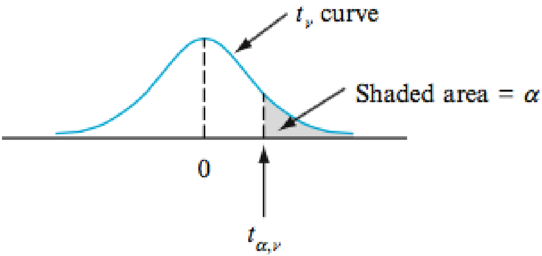
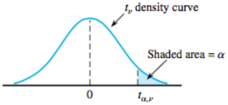


Figure 8.7 A pictorial definition of $t_{\alpha,\nu}$

$\alpha \to t$

Table A.5 Critical Values for *t* Distributions



				α			
ν	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745

Prediction intervals

Settings

- We have available a random sample $X_1, X_2, \ldots, X_n$ from a normal population distribution
- We wish to predict the value of X_{n+1}, a single future observation.

This is a much more difficult problem than the problem of estimating μ

- When $n \to \infty$, $\bar{X} \to \mu$
- Even when we know μ, X_{n+1} is still random

Problem

Let $X_1, X_2, \ldots, X_n$ be a sample from a normal population distribution $\mathcal{N}(\mu, \sigma)$ and X_{n+1} be an independent sample from the same distribution.

- Compute $E[\bar{X} - X_{n+1}]$ in terms of μ, σ, n
- Compute $Var[\bar{X} - X_{n+1}]$ in terms of μ, σ, n
- What is the distribution of $\bar{X} - X_{n+1}$?

Principle

If σ is known

$$\frac{\bar{X} - X_{n+1}}{\sigma\sqrt{1 + \frac{1}{n}}}$$

follows the standard normal distribution $\mathcal{N}(0, 1)$.

Principle

$$T = \frac{\overline{X} - X_{n+1}}{S\sqrt{1 + \frac{1}{n}}} \sim t \text{ distribution with } n - 1 \text{ df}$$

Prediction intervals

A prediction interval (PI) for a single observation to be selected from a normal population distribution is

$$\bar{x} \pm t_{\alpha/2,n-1} \cdot s\sqrt{1 + \frac{1}{n}} \qquad (8.16)$$

The *prediction level* is $100(1-\alpha)\%$.

Example 4b

Example

Let X be the amount of butterfat in pounds produced by a typical cow during a 305-day milk production period between her first and second calves. Assume that the distribution of X is $N(\mu, \sigma^2)$. To estimate μ, a farmer measured the butterfat production for n = 20 cows and obtained the following data:

481	537	513	583	453	510	570	500	457	555
618	327	350	643	499	421	505	637	599	392

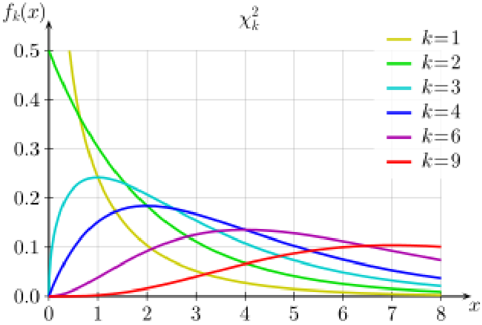
Construct a 90% prediction interval for μ.

Section 6.4: Distributions based on a normal random sample

- The Chi-squared distribution
- The t distribution
- The F Distribution

The pdf of a Chi-squared distribution with degree of freedom ν, denoted by χ^2_ν, is

$$f(x) = \begin{cases} \frac{1}{2^{1/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



Why is Chi-squared useful?

Proposition

If Z has standard normal distribution $\mathcal{Z}(0,1)$ and $X = Z^2$, then X has Chi-squared distribution with 1 degree of freedom, i.e. $X \sim \chi^2_1$ distribution.

Proposition

If $X_1 \sim \chi^2_{\nu_1}$, $X_2 \sim \chi^2_{\nu_2}$ and they are independent, then

$$X_1 + X_2 \sim \chi^2_{\nu_1+\nu_2}$$

Why is Chi-squared useful?

Proposition

If $Z_1, Z_2, \ldots, Z_n$ are independent and each has the standard normal distribution, then

$$Z_1^2 + Z_2^2 + \ldots + Z_n^2 \sim \chi_n^2$$

Why is Chi-squared useful?

If $X_1, X_2, \ldots, X_n$ is a random sample from the normal distribution $\mathcal{N}(\mu, \sigma^2)$, then

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

t distributions

Let Z be a standard normal rv and let X be a χ^2_ν rv independent of Z. Then the t distribution with degrees of freedom ν is defined to be the distribution of the ratio

$$T = \frac{Z}{\sqrt{X/\nu}}$$

t distributions

When $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ, the rv

$$\frac{\bar{X}-\mu}{S/\sqrt{n}}$$

has the t distribution with $n-1$ degree of freedom (df).
Hint:

$$T = \frac{Z}{\sqrt{X/\nu}} \qquad (n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

and

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \cdot \frac{1}{\sqrt{(n-1)\frac{S^2}{\sigma^2}/(n-1)}}.$$

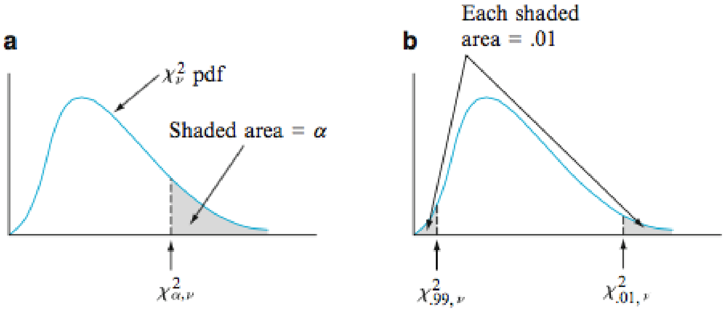
CIs for variance and standard deviation

Why is Chi-squared useful?

If $X_1, X_2, \dots, X_n$ is a random sample from the normal distribution $\mathcal{N}(\mu, \sigma^2)$, then

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Important: Chi-squared distribution are not symmetric



CIs for standard deviation

We have

$$P\left(\chi^2_{1-\alpha/2,n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

Play around with these inequalities:

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

CIs for standard deviation

A $100(1-\alpha)\%$ confidence interval for the variance σ^2 of a normal population has lower limit

$$(n-1)s^2/\chi^2_{\alpha/2,n-1}$$

and upper limit

$$(n-1)s^2/\chi^2_{1-\alpha/2,n-1}$$

A confidence interval for σ has lower and upper limits that are the square roots of the corresponding limits in the interval for σ^2.

Practice problems

Example 1

Problem

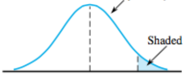
Here are the alcohol percentages for a sample of 16 beers:

4.68	4.13	4.80	4.63	5.08	5.79	6.29	6.79
4.93	4.25	5.70	4.74	5.88	6.77	6.04	4.95

(a) Assume the distribution is normal, construct the 95% confidence interval for the population mean.

$\alpha \to t$

Table A.5 Critical Values for t Distributions



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23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745

Example 1b

Problem

Here are the alcohol percentages for a sample of 16 beers:

4.68	4.13	4.80	4.63	5.08	5.79	6.29	6.79
4.93	4.25	5.70	4.74	5.88	6.77	6.04	4.95

(b) Assume the distribution is normal, construct the 95% lower confidence bound for the population mean.

Example 1c

Problem

Here are the alcohol percentages for a sample of 16 beers:

4.68	4.13	4.80	4.63	5.08	5.79	6.29	6.79
4.93	4.25	5.70	4.74	5.88	6.77	6.04	4.95

(b) Assume that another beer is sampled from the same distribution, construct the 95% prediction interval for the alcohol percentages of that beer.

Example 2

Problem

Suppose that against a certain opponent, the number of points a basketball team scores is normally distributed with unknown mean μ and unknown variance σ^2. Suppose that over the course of the last 10 games, the team scored the following points:

$$59, \quad 62, \quad 59, \quad 74, \quad 70, \quad 61, \quad 62, \quad 66, \quad 62, \quad 75$$

- *Construct a 95% confidence interval for μ.*
- *Now suppose that you learn that $\sigma^2 = 25$. Construct a 95% confidence interval for μ. How does this compare to the interval in (a)?*

Example 3

Problem

A study of the ability of individuals to walk in a straight line reported the accompanying data on cadence (strides per second) for a sample of $n = 20$ randomly selected healthy men. Assuming that the distribution is normal:

- *Calculate a 95% confidence interval for population mean cadence*
- *Calculate and interpret a 95% prediction interval for the cadence of a single individual randomly selected from this population.*