MATH 450: Mathematical statistics

November 10th, 2020

Lecture 19: Tests about a population mean

Overview

Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	Chapter 7: Point Estimation
Week 7	Chapter 8: Confidence Intervals
Week 10	**Chapter 9: Tests of Hypotheses**
Week 12	Chapter 10: Two-sample testing
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Overview

Hypothesis testing

Hypothesis testing

In any hypothesis-testing problem, there are two contradictory hypotheses under consideration

- The null hypothesis, denoted by H_0, is the claim that is initially assumed to be true
- The alternative hypothesis, denoted by H_a, is the assertion that is contradictory to H_0.

Implicit rules (of this chapter)

- H_0 will always be stated as an equality claim.
- If θ denotes the parameter of interest, the null hypothesis will have the form
$$H_0 : \theta = \theta_0$$
- θ_0 is a specified number called the *null value*
- The alternative hypothesis will be either:
  - $H_a : \theta > \theta_0$
  - $H_a : \theta < \theta_0$
  - $H_a : \theta \neq \theta_0$

Test procedures

A test procedure is specified by the following:

- A test statistic T: a function of the sample data on which the decision (reject H_0 or do not reject H_0) is to be based
- A rejection region $\mathcal{R}$: the set of all test statistic values for which H_0 will be rejected

The null hypothesis will then be rejected if and only if the observed or computed test statistic value falls in the rejection region, i.e., $T \in \mathcal{R}$

Type I and Type II errors

- A type I error consists of rejecting the null hypothesis H_0 when it is true
- A type II error involves not rejecting H_0 when H_0 is false.

Example 2: Type I error

- It is believed that drying times of an additive follows $\mathcal{N}(\mu, 9^2)$.
- Test of hypotheses:

$$H_0 : \mu = 75$$
$$H_a : \mu < 75$$

- $n = 25$. Rule: If $\bar{x} \leq 72$, reject H_0.
- Question: What is the probability of type I error?

$$\begin{aligned}
\alpha &= P[\text{Type I error}] \\
&= P[H_0 \text{ is rejected while it is true}] \\
&= P[\bar{X} \leq 72 \text{ while } \mu = 75] \\
&= P[\bar{X} \leq 72 \text{ while } \bar{X} \sim \mathcal{N}(75, 1.8^2)] = 0.0475
\end{aligned}$$

Example 2: Type II error

- It is believed that drying times of an additive follows $\mathcal{N}(\mu, 9^2)$.
- Test of hypotheses:

$$H_0 : \mu = 75$$
$$H_a : \mu < 75$$

- Experimental data is to consist of drying times from $n = 25$ test specimens: $X_1, X_2, \ldots, X_{25}$.
- New rule:
  - Compute $\bar{x}$
  - If $\bar{x} \leq 72$, reject H_0.
- Question: What are $\beta(70)$?

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Type II error

- Test of hypotheses:

$$H_0 : \mu = 75$$
$$H_a : \mu < 75$$

- $n = 25$. Rule: If $\bar{x} \leq 72$, reject H_0.

$$\begin{aligned}\beta(70) &= P[\text{Type II error when } \mu = 70] \\ &= P[H_0 \text{ is not rejected while it is false because } \mu = 70] \\ &= P[\bar{X} > 72 \text{ while } \mu = 70] \\ &= P[\bar{X} > 72 \text{ while } \bar{X} \sim \mathcal{N}(70, 1.8^2)] = 0.1335\end{aligned}$$

Example 2b

- Test of hypotheses:

$$H_0 : \mu = 75$$
$$H_a : \mu < 75$$

- $n = 25$. New rule: If $\bar{x} \leq c$, reject H_0.
- Find the value of c to make the probability of making Type I error equal to 0.1

Rejection region

$$\begin{aligned}
\alpha &= P[\text{Type I error}] \\
&= P[H_0 \text{ is rejected while it is true}] \\
&= P[\bar{X} \le c \text{ while } \bar{X} \sim \mathcal{N}(75, 1.8^2)] \\
&= P\left[\frac{\bar{X} - 75}{1.8} \le \frac{c - 75}{1.8}\right]
\end{aligned}$$

- Rejection rule: $\bar{x} \le 75 - 1.8 z_\alpha$
- To make it simpler, define $z = (\bar{x} - 75)/(1.8)$, then the rule is

$$z \le -z_\alpha$$

Example 2c

- If we want to test

$$H_0 : \mu = 75$$
$$H_a : \mu \neq 75$$

- $n = 25$. Rule: If

$$\bar{x} \leq a \quad \text{or} \quad \bar{x} \geq b$$

  reject H_0.

- Find the value of a, b to make $\alpha = 0.1$

$\alpha - \beta$ compromise

Proposition

Suppose an experiment and a sample size are fixed and a test statistic is chosen. Then decreasing the size of the rejection region to obtain a smaller value of α results in a larger value of β for any particular parameter value consistent with H_a.

Significance level

The approach adhered to by most statistical practitioners is

- specify the largest value of α that can be tolerated
- find a rejection region having that value of α rather than anything smaller
- α: the *significance level* of the test
- the corresponding test procedure is called a *level* α *test*

Hypothesis testing for one parameter

1. Identify the parameter of interest
2. Determine the null value and state the null hypothesis
3. State the appropriate alternative hypothesis
4. Give the formula for the test statistic
5. State the rejection region for the selected significance level α
6. Compute statistic value from data
7. Decide whether H_0 should be rejected and state this conclusion in the problem context

Normal population with known σ

Test about a population mean

- Null hypothesis

$$H_0 : \mu = \mu_0$$

- The alternative hypothesis will be either:
  - $H_a : \mu > \mu_0$
  - $H_a : \mu < \mu_0$
  - $H_a : \mu \neq \mu_0$

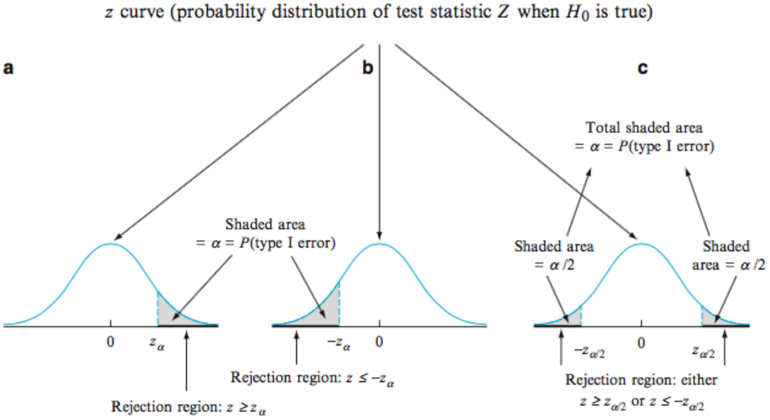
Normal population with known σ

Null hypothesis: $\mu = \mu_0$
Test statistic:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Alternative Hypothesis	Rejection Region for Level α Test
H_a: $\mu > \mu_0$	$z \geq z_\alpha$ (upper-tailed test)
H_a: $\mu < \mu_0$	$z \leq -z_\alpha$ (lower-tailed test)
H_a: $\mu \neq \mu_0$	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

General rule



Example

Problem

A manufacturer of sprinkler systems used for fire protection in office buildings claims that the true average system-activation temperature is $130^\circ F$*. A sample of* $n = 9$ *systems, when tested, yields a sample average activation temperature of* $131.08^\circ F$*.*

If the distribution of activation times is normal with standard deviation $1.5^\circ F$*, does the data contradict the manufacturer's claim at significance level* $\alpha = 0.01$*?*

Solution

- Parameter of interest: $\mu =$ true average activation temperature
- Hypotheses

$$H_0 : \mu = 130$$
$$H_a : \mu \neq 130$$

- Test statistic:

$$z = \frac{\bar{x} - 130}{1.5/\sqrt{n}}$$

- Rejection region: either $z \leq -z_{0.005}$ or $z \geq z_{0.005} = 2.58$
- Substituting $\bar{x} = 131.08$, $n = 25 \rightarrow z = 2.16$.
- Note that $-2.58 < 2.16 < 2.58$. We fail to reject H_0 at significance level 0.01.
- The data does not give strong support to the claim that the true average differs from the design value.