MATH 450: Mathematical statistics

December 3rd, 2019

Lecture 24: The paired t-test

Overview

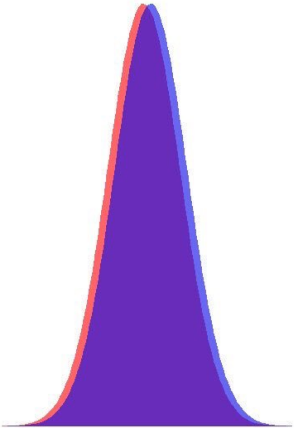
Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	Chapter 7: Point Estimation
Week 7	Chapter 8: Confidence Intervals
Week 10	Chapter 9: Tests of Hypotheses
Week 12	**Chapter 10: Two-sample testing**

10.1 Difference between two population means
- z-test
- confidence intervals

10.2 The two-sample t test and confidence interval

10.3 **Analysis of paired data**

Difference between two population means



- Testing:

$$H_0 : \mu_1 = \mu_2$$
$$H_a : \mu_1 > \mu_2$$

- Works well, even if $|\mu_1 - \mu_2| << \sigma_1, \sigma_2$

Settings

- Last lecture: independent samples

Assumption

1. *$X_1, X_2, \ldots, X_m$ is a random sample from a population with mean μ_1 and variance σ_1^2.*
2. *$Y_1, Y_2, \ldots, Y_n$ is a random sample from a population with mean μ_2 and variance σ_2^2.*
3. *The X and Y samples are independent of each other.*

- This lecture: paired-sample test

Independent setting

Remember Chapter 8?

- Section 8.1
  - Normal distribution
  - σ is known
- Section 8.2
  - ~~Normal distribution~~
    $\rightarrow$ Using Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ needs $n > 40$
- Section 8.3
  - Normal distribution
  - ~~σ is known~~
  - n is small

  $\rightarrow$ Introducing t-distribution

Properties of $\bar{X} - \bar{Y}$

Proposition

The expected value of $\overline{X} - \overline{Y}$ is $\mu_1 - \mu_2$, so $\overline{X} - \overline{Y}$ is an unbiased estimator of $\mu_1 - \mu_2$. The standard deviation of $\overline{X} - \overline{Y}$ is

$$\sigma_{\overline{X}-\overline{Y}} = \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

Confidence intervals: Normal distributions with known variances

When both population distributions are normal, standardizing $\overline{X} - \overline{Y}$ gives a random variable Z with a standard normal distribution. Since the area under the z curve between $-z_{\alpha/2}$ and $z_{\alpha/2}$ is $1 - \alpha$, it follows that

$$P\left(-z_{\alpha/2} < \frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} < z_{\alpha/2}\right) = 1 - \alpha$$

Manipulation of the inequalities inside the parentheses to isolate $\mu_1 - \mu_2$ yields the equivalent probability statement

$$P\left(\overline{X} - \overline{Y} - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}} < \mu_1 - \mu_2 < \overline{X} - \overline{Y} + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right) = 1 - \alpha$$

Testing the difference between two population means

Proposition

Null hypothesis: H_0: $\mu_1 - \mu_2 = \Delta_0$

Test statistic value: $z = \dfrac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}}$

Alternative Hypothesis	**Rejection Region for Level α Test**
H_a: $\mu_1 - \mu_2 > \Delta_0$	$z \geq z_\alpha$ (upper-tailed test)
H_a: $\mu_1 - \mu_2 < \Delta_0$	$z \leq -z_\alpha$ (lower-tailed test)
H_a: $\mu_1 - \mu_2 \neq \Delta_0$	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

Large-sample tests/confidence intervals

- Central Limit Theorem: $\bar{X}$ and $\bar{Y}$ are approximately normal when $n > 30 \rightarrow$ so is $\bar{X} - \bar{Y}$. Thus

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$$

  is approximately standard normal
- When n is sufficiently large $S_1 \approx \sigma_1$ and $S_2 \approx \sigma_2$
- Conclusion:

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

  is approximately standard normal when n is sufficiently large

If $m, n > 40$, we can ignore the normal assumption and replace σ by S

Large-sample CIs

Proposition

Provided that m and n are both large, a CI for $\mu_1 - \mu_2$ *with a confidence level of approximately* $100(1-\alpha)\%$ *is*

$$\bar{x} - \bar{y} \ \pm \ z_{\alpha/2}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}$$

where $-$*gives the lower limit and* $+$ *the upper limit of the interval. An upper or lower confidence bound can also be calculated by retaining the appropriate sign and replacing* $z_{\alpha/2}$ *by* z_α*.*

Proposition

Use of the test statistic value

$$z = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

along with the previously stated upper-, lower-, and two-tailed rejection regions based on z critical values gives large-sample tests whose significance levels are approximately α. These tests are usually appropriate if both $m > 40$ and $n > 40$. A P-value is computed exactly as it was for our earlier z tests.

2-sample t-test: degree of freedom

THEOREM When the population distributions are both normal, the standardized variable

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} \qquad (10.2)$$

has approximately a t distribution with df ν estimated from the data by

$$\nu = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}} = \frac{\left[(se_1)^2 + (se_2)^2\right]^2}{\frac{(se_1)^4}{m-1} + \frac{(se_2)^4}{n-1}}$$

where

$$se_1 = \frac{s_1}{\sqrt{m}} \qquad se_2 = \frac{s_2}{\sqrt{n}}$$

(round ν down to the nearest integer).

The **two-sample t confidence interval for $\mu_1 - \mu_2$** with confidence level $100(1-\alpha)\%$ is then

$$\bar{x} - \bar{y} \pm t_{\alpha/2,v}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}$$

A one-sided confidence bound can be calculated as described earlier.

2-sample t procedures

The **two-sample t test** for testing H_0: $\mu_1 - \mu_2 = \Delta_0$ is as follows:

$$\text{Test statistic value: } t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

Alternative Hypothesis	**Rejection Region for Approximate Level α Test**
H_a: $\mu_1 - \mu_2 > \Delta_0$	$t \geq t_{\alpha,\nu}$ (upper-tailed test)
H_a: $\mu_1 - \mu_2 < \Delta_0$	$t \leq -t_{\alpha,\nu}$ (lower-tailed test)
H_a: $\mu_1 - \mu_2 \neq \Delta_0$	either $t \geq t_{\alpha/2,\nu}$ or $t \leq -t_{\alpha/2,\nu}$ (two-tailed test)

A P-value can be computed as described in Section 9.4 for the one-sample t test.

The paired samples setting

1. There is only one set of n individuals or experimental objects
2. Two observations are made on each individual or object

Example

Example

Consider two scenarios:

A. Insulin rate is measured on 30 patients before and after a medical treatment.

B. Insulin rate is measured on 30 patients receiving a placebo and 30 other patients receiving a medical treatment.

Notes

- In the independent case, we construct the statistics by looking at the distribution of

$$D = \bar{X} - \bar{Y}$$

which has

$$E[\bar{X} - \bar{Y}] = \mu_1 - \mu_2, \qquad Var[\bar{X} - \bar{Y}] = Var(\bar{X}) + Var(\bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$$

- With paired data, the X and Y observations within each pair are not independent, so $\bar{X}$ and $\bar{Y}$ are not independent of each other $\rightarrow$ the computation of the variance is in valid $\rightarrow$ could not use the old formulas

The paired t-test

- Because different pairs are independent, the D_i's are independent of each other
- We also have

$$E[D] = E[X - Y] = E[X] - E[Y] = \mu_1 - \mu_2 = \mu_D$$

- Testing about $\mu_1 - \mu_2$ is just the same as testing about μ_D
- Idea: to test hypotheses about $\mu_1 - \mu_2$ when data is paired:
  1. form the differences $D_1, D_2, \ldots, D_n$
  2. carry out a one-sample t-test (based on $n - 1$ df) on the differences.

Settings

Assumption

1. *The data consists of n independently selected pairs of independently normally distributed random variables* $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ *with* $E(X_i) = \mu_1$ *and* $E(Y_i) = \mu_2$.
2. *Let*

$$D_1 = X_1 - Y_1, \qquad D_2 = X_2 - Y_2, \dots, \qquad D_n = X_n - Y_n,$$

so the D_i*'s are the differences within pairs.*

Confidence intervals

- A t confidence interval for for $\mu_D = \mu_1 - \mu_2$ can be constructed based on the fact that

$$T = \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}}$$

  follows the t distribution with degree of freedom $n - 1$.
- The CI for μ_D is

$$\bar{d} \pm t_{\alpha/2,n-1}\frac{s_D}{\sqrt{n}}$$

- A one-sided confidence bound results from retaining the relevant sign and replacing $t_{\alpha/2,n-1}$ by $t_{\alpha,n-1}$.

THE PAIRED *t* TEST

Null hypothesis: H_0: $\mu_D = \Delta_0$ (where $D = X - Y$ is the difference between the first and second observations within a pair, and $\mu_D = \mu_1 - \mu_2$)

Test statistic value: $t = \dfrac{\overline{d} - \Delta_0}{s_D/\sqrt{n}}$ (where $\overline{d}$ and s_D are the sample mean and standard deviation, respectively, of the d_i's)

Alternative Hypothesis	**Rejection Region for Level α Test**
H_a: $\mu_D > \Delta_0$	$t \geq t_{\alpha,n-1}$
H_a: $\mu_D < \Delta_0$	$t \leq -t_{\alpha,n-1}$
H_a: $\mu_D \neq \Delta_0$	either $t \geq t_{\alpha/2,n-1}$ or $t \leq -t_{\alpha/2,n-1}$

A P-value can be calculated as was done for earlier t tests.

Example

Example

Consider two scenarios:

A. Insulin rate is measured on 30 patients before and after a medical treatment.

B. Insulin rate is measured on 30 patients receiving a placebo and 30 other patients receiving a medical treatment.

What type of test should be used in each cased: paired or unpaired?

Example

Example

Suppose we have a new synthetic material for making soles for shoes. We'd like to compare the new material with leather – using some energetic kids who are willing to wear test shoes and return them after a time for our study. Consider two scenarios:

A. Giving 50 kids synthetic sole shoes and 50 kids leather shoes and then collect them back, comparing the average wear in each group

B. Give each of a random sample of 50 kids one shoe made by the new synthetic materials and one shoe made with leather

What type of test should be used in each cased: paired or unpaired?

Example

Consider an experiment in which each of 13 workers was provided with both a conventional shovel and a shovel whose blade was perforated with small holes. The following data on stable energy expenditure is provided:

Worker:	1	2	3	4	5	6	7
Conventional:	.0011	.0014	.0018	.0022	.0010	.0016	.0028
Perforated:	.0011	.0010	.0019	.0013	.0011	.0017	.0024

Worker:	8	9	10	11	12	13
Conventional:	.0020	.0015	.0014	.0023	.0017	.0020
Perforated:	.0020	.0013	.0013	.0017	.0015	.0013

Calculate a confidence interval at the 95% confidence level for the true average difference between energy expenditure for the conventional shovel and the perforated shovel (assuming that the differences follow normal distribution).

Example

Consider an experiment in which each of 13 workers was provided with both a conventional shovel and a shovel whose blade was perforated with small holes. The following data on stable energy expenditure is provided:

Worker:	1	2	3	4	5	6	7
Conventional:	.0011	.0014	.0018	.0022	.0010	.0016	.0028
Perforated:	.0011	.0010	.0019	.0013	.0011	.0017	.0024

Worker:	8	9	10	11	12	13
Conventional:	.0020	.0015	.0014	.0023	.0017	.0020
Perforated:	.0020	.0013	.0013	.0017	.0015	.0013

Carry out a test of hypotheses at significance level .05 to see if true average energy expenditure using the conventional shovel exceeds that using the perforated shovel; include a P-value in your analysis.

t-table

t \ ν	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1.6	.178	.125	.104	.092	.085	.080	.077	.074	.072	.070	.069	.068	.067	.065	.065	.065	.064	.064
1.7	.169	.116	.094	.082	.075	.070	.065	.064	.062	.060	.059	.057	.056	.055	.055	.054	.054	.053
1.8	.161	.107	.085	.073	.066	.061	.057	.055	.053	.051	.050	.049	.048	.046	.046	.045	.045	.044
1.9	.154	.099	.077	.065	.058	.053	.050	.047	.045	.043	.042	.041	.040	.038	.038	.038	.037	.037
2.0	.148	.092	.070	.058	.051	.046	.043	.040	.038	.037	.035	.034	.033	.032	.032	.031	.031	.030
2.1	.141	.085	.063	.052	.045	.040	.037	.034	.033	.031	.030	.029	.028	.027	.027	.026	.025	.025
2.2	.136	.079	.058	.046	.040	.035	.032	.029	.028	.026	.025	.024	.023	.022	.022	.021	.021	.021
2.3	.131	.074	.052	.041	.035	.031	.027	.025	.023	.022	.021	.020	.019	.018	.018	.018	.017	.017
2.4	.126	.069	.048	.037	.031	.027	.024	.022	.020	.019	.018	.017	.016	.015	.015	.014	.014	.014
2.5	.121	.065	.044	.033	.027	.023	.020	.018	.017	.016	.015	.014	.013	.012	.012	.012	.011	.011
2.6	.117	.061	.040	.030	.024	.020	.018	.016	.014	.013	.012	.012	.011	.010	.010	.010	.009	.009
2.7	.113	.057	.037	.027	.021	.018	.015	.014	.012	.011	.010	.010	.009	.008	.008	.008	.008	.007
2.8	.109	.054	.034	.024	.019	.016	.013	.012	.010	.009	.009	.008	.008	.007	.007	.006	.006	.006
2.9	.106	.051	.031	.022	.017	.014	.011	.010	.009	.008	.007	.007	.006	.005	.005	.005	.005	.005
3.0	.102	.048	.029	.020	.015	.012	.010	.009	.007	.007	.006	.006	.005	.004	.004	.004	.004	.004

Example

A physician claims that an experimental medication *increases* an individuals heart rate. Twelve test subjects are randomly selected, and the heart rate of each is measured. The subjects are then injected with the medication and, after 1 hour, the heart rate of each is measured again. The results are show below.

Before	72	81	76	74	75	80	68	75	78	76	74	77
After	73	80	79	76	76	80	74	77	75	74	76	78

We assume that both samples were selected from normal distributions.

(a) (20 points) Carry out a test of hypotheses with significance level $\alpha = 0.05$ to decide whether we should reject the physician's claim. Provide the P-value of the test.

(20 points) Another physician claims that the experimental medication above does *change* the heart rate. Do we have enough evidence to reject his claim, using the same significance level $\alpha = 0.05$?