MATH 205: Statistical methods

September 22nd, 2021

Lecture 6: Conditional probability

Tentative schedule

Date	Theme/Topic	Labs	Assignments
Sep 1	Syllabus		
Sep 8	Chapter 1: Describing dataset	Section 2: Handling data	
Sep 13 - 15	Chapter 2: Looking at Relationships	Section 3: Univariate data	
Sep 20-22	Chapter 3: Basic Ideas in Probability	Section 4: Bivariate Data	Homework 1 (due 09/22)
Sep 27-29	Chapters 3-4	Section 4: Correlation	
Oct 4-6	Chapter 4: Random variables and expectations	Section 6: Random data	Homework 2 (due 10/06)
Oct 11-13	Chapter 5: Useful distributions	Section 7: The central limit theorem	
Oct 18-20	Chapter 6: Samples and populations	Section 9: Confidence interval estimation	Homework 3 (due 10/20)
Oct 25-27	Review and midterm exam		Midterm: Oct 27 (lecture), Oct 25-27 (labs)
Nov 1-3	Chapter 7: The significance of evidence	Section 10: Hypothesis testing	
Nov 8-10	Goodness of Fit	Section 12: Goodness of Fit	Homework 4 (due 11/10)
Nov 15-17	Linear Regression	Section 13: Linear regression	
Nov 22-24	Thanksgiving break		
Nov 29 - Dec 1	One-Way Analysis of Variance	Section 15: Analysis of variance	Homework 5 (due 12/01)
Dec 6-8	Selected topics + Review		
Exam week			

Think like a Bayesian statistician

- Given an experiment and a sample space, we can define many different probabilities
- Experiment: tossing a coin, $\Omega = \{H, T\}$
- If you believe the coin is fair:

$$P(\emptyset) = 0, \quad P(\{H\}) = 0.5, \quad P(\{T\}) = 0.5, \quad P(\{H, T\}) = 1.$$

- If you do not, then maybe

$$P(\emptyset) = 0, \quad P(\{H\}) = 0.7, \quad P(\{T\}) = 0.3, \quad P(\{H, T\}) = 1.$$

Example: the JEDI contract (2019)

- is a large United States Department of Defense cloud computing contract that worths 10 billion.
- three outcomes: All-others (1), Microsoft (2), and Amazon (3)

$$\Omega = \{1, 2, 3\}$$

- Let's say, originally, we believed that

$$P(1) = 1/5, \quad P(2) = 2/5, \quad P(3) = 2/5$$

Example: the JEDI contract

Amazon Accuses Trump of 'Improper Pressure' on JEDI Contract

In a legal complaint, Amazon said the president had attacked it behind the scenes to harm its C.E.O., Jeff Bezos, "his perceived political enemy."

Amazon had been considered the front-runner for the Joint Enterprise Defense Infrastructure project, known as JEDI. Mark Lennihan/Associated Press

Probability updated with new information

- Let's say, originally, we believe that

$$P(1) = 1/5, \quad P(2) = 2/5, \quad P(3) = 2/5$$

- Suppose that we learn that the outcome is 1 or 2 (This means, the event $A = \{1, 2\}$ happens)
- How should we adapt our model?

Conditional probability

- Denote the new probability by $\tilde{P}$
- We know $\tilde{P}(3) = 0$, and $\tilde{P}(1) + \tilde{P}(2) = 1$
- The new information should not alter the relative chances of 1 and 2
- We can obtain these by setting

$$\tilde{P}(1) = \frac{P(1)}{P(A)}, \quad \tilde{P}(2) = \frac{P(2)}{P(A)}$$

Conditional probability

Definition

Let $P(A) > 0$, the conditional probability of B given A, denoted by $P(B|A)$, is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Conditional probability

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Let $P(A) > 0$, the conditional probability of B given A, denoted by $P(B|A)$, is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

- This definition does not make sense if $P(A) = 0$ (we will learn how to deal with this later)
- The newly defined probability satisfies the 3 rules of probability

Properties

- Rearrange the definition

$$P(B \cap A) = P(B|A)P(A)$$

$\rightarrow$ sometimes called the **law of multiplication**.

- **Bayes' rule**

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Remember $P(A) = P(A \cap B) + P(A \cap B^c)$? We deduce that

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$\rightarrow$ sometimes called the **law of total probability**.

Law of multiplication

Useful Facts 3.3 (Conditional Probability Formulas)
You should remember the following formulas:

- $P(\mathcal{B}|\mathcal{A}) = \dfrac{P(\mathcal{A}|\mathcal{B})P(\mathcal{B})}{P(\mathcal{A})}$
- $P(\mathcal{A}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c)$
- Assume (a) $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ for $i \neq j$ and (b) $\mathcal{A} \cap (\cup_i \mathcal{B}_i) = \mathcal{A}$; then $P(\mathcal{A}) = \sum_i P(\mathcal{A}|\mathcal{B}_i)P(\mathcal{B}_i)$

Example 1

Problem

We throw two fair six-sided dice. What is the conditional probability that the sum of spots on both dice is greater than six, conditioned on the event that the first die comes up five?

Example 2

Problem

We flip the coin 3 times and encodes the outcomes of a flip as 0 for head and 1 for tails.

- *What is the probability of getting two heads out of three?*
- *Suppose the first flip is revealed to be head, what is the probability of getting two heads out of three?*

Example 3

Problem

Suppose an urn contains 8 red and 4 white balls. Draw two balls without replacement. What is the probability that both are red?

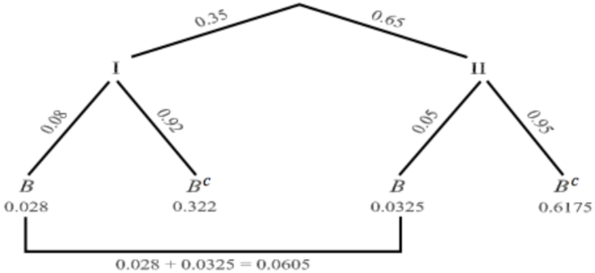
Example 4

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Example 4



0.35
0.65
I
II
0.08
0.92
0.05
0.95
B
0.028
B^c
0.322
B
0.0325
B^c
0.6175
0.028 + 0.0325 = 0.0605

Example 4b

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Question: Assuming that a randomly selected car broke down, what is the probability that this car is from agency I?