MATH 205: Statistical methods

September 27th, 2021

Lecture 7: Random variables

Announcements

- Homework 1 due this Wednesday
- Homework 2 was uploaded to the course webpage, due next Wednesday
- There will be a quiz in class this Wednesday. The quiz covers the materials of Chapter 3
- Countdown to midterm exam: 30 days

Tentative schedule

Date	Theme/Topic	Labs	Assignments
Sep 1	Syllabus		
Sep 8	Chapter 1: Describing dataset	Section 2: Handling data	
Sep 13 - 15	Chapter 2: Looking at Relationships	Section 3: Univariate data	
Sep 20-22	Chapter 3: Basic Ideas in Probability	Section 4: Bivariate Data	Homework 1 (due 09/22)
Sep 27-29	Chapters 3-4	Section 4: Correlation	
Oct 4-6	Chapter 4: Random variables and expectations	Section 6: Random data	Homework 2 (due 10/06)
Oct 11-13	Chapter 5: Useful distributions	Section 7: The central limit theorem	
Oct 18-20	Chapter 6: Samples and populations	Section 9: Confidence interval estimation	Homework 3 (due 10/20)
Oct 25-27	Review and midterm exam		Midterm: Oct 27 (lecture), Oct 25-27 (labs)
Nov 1-3	Chapter 7: The significance of evidence	Section 10: Hypothesis testing	
Nov 8-10	Goodness of Fit	Section 12: Goodness of Fit	Homework 4 (due 11/10)
Nov 15-17	Linear Regression	Section 13: Linear regression	
Nov 22-24	Thanksgiving break		
Nov 29 - Dec 1	One-Way Analysis of Variance	Section 15: Analysis of variance	Homework 5 (due 12/01)
Dec 6-8	Selected topics + Review		
Exam week			

Review: Basic ideas in probability

Basic ideas in probability

3.1 Sample space, events

3.2 Probability

3.3 Independence

3.4 Conditional probability

Sample space and events

1. An experiment: is any action, process, or phenomenon whose outcome is subject to uncertainty
2. An outcome: is a result of an experiment
   Each run of the experiment results in one outcome
3. A sample space: is the set of all possible outcomes of an experiment
4. An event: is a subset of the sample space.
   An event occurs when one of the outcomes that belong to it occurs

Basic properties of probability

Useful Facts 3.1 (Basic Properties of the Probability Events)
We have

- The probability of every event is between zero and one; in equations

$$0 \leq P(\mathcal{A}) \leq 1$$

  for any event $\mathcal{A}$.
- Every experiment has an outcome; in equations,

$$P(\Omega) = 1.$$

- The probability of disjoint events is additive; writing this in equations requires some notation. Assume that we have a collection of events $\mathcal{A}_i$, indexed by i. We require that these have the property $\mathcal{A}_i \cap \mathcal{A}_j = \varnothing$ when $i \neq j$. This means that there is no outcome that appears in more than one $\mathcal{A}_i$. In turn, if we interpret probability as relative frequency, we must have that

$$P(\cup_i \mathcal{A}_i) = \sum_i P(\mathcal{A}_i)$$

Advanced properties of probability

Useful Facts 3.2 (Properties of the Probability of Events)

- $P(\mathcal{A}^c) = 1 - P(\mathcal{A})$
- $P(\varnothing) = 0$
- $P(\mathcal{A} - \mathcal{B}) = P(\mathcal{A}) - P(\mathcal{A} \cap \mathcal{B})$
- $P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A} \cap \mathcal{B})$

- If $A \subset B$, then $P(A) \leq P(B)$.
- For any events A, B

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

Independence

Definition

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Conditional probability

Definition

Let $P(A) > 0$, the conditional probability of B given A, denoted by $P(B|A)$, is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Properties of Conditional probability

- Law of multiplication

$$P(B \cap A) = P(B|A)P(A)$$

- **Bayes' rule**

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Law of total probability

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

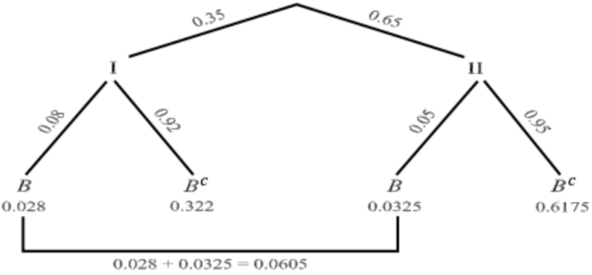
Example 4

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Example 4



0.35
0.65
I
II
0.08
0.92
0.05
0.95
B
0.028
B^c
0.322
B
0.0325
B^c
0.6175
0.028 + 0.0325 = 0.0605

Example 4b

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Problem

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

Question: Assuming that a randomly selected car broke down, what is the probability that this car is from agency I?

Example 5

Problem

You have a blood test for a rare disease that occurs by chance in 1 person in 100,000. If you have the disease, the test will report that you do with probability 0.95 (and that you do not with probability 0.05). If you do not have the disease, the test will report a false positive with probability 1e-3. If the test says you do have the disease, what is the probability that you actually have the disease?

Conditional probability for independent events

Useful Facts 3.4 (Conditional Probability for Independent Events)
If two events $\mathcal{A}$ and $\mathcal{B}$ are independent, then

$$P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$$

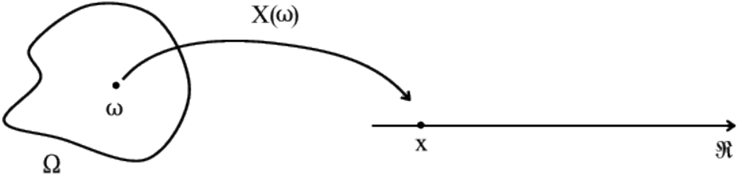
and

$$P(\mathcal{B}|\mathcal{A}) = P(\mathcal{B}).$$

Chapter 4: Random variables and expectations

Random variables and expectations

Random variable



Definition

Given an experiment with sample space Ω, set of events $\mathcal{F}$ and probability P, a real-valued function $X : \Omega \to \mathbb{R}$ is called a random variable of the experiment.

(This means that for any outcome ω there is a number $X(\omega)$)

Random variable: examples

Example (Numbers from Coins)

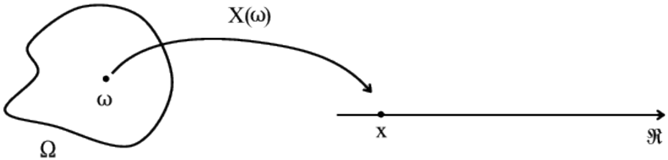
We flip a coin. Whenever the coin comes up heads, we report 1; when it comes up tails, we report 0. This is a random variable.

Random variable: examples

Example (Numbers from Coins)

We flip a coin 32 times. We record a 1 when it comes up heads, and when it comes up tails, we record a 0. This produces a 32-bit random number, which is a random variable.

Random variables



Notations:

- random variables are denoted by uppercase letters (e.g., X);
- the observed values of the random variables are denoted by lowercase letters (e.g., x)

Discrete random variable

Definition

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

Note: A set A is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers, i.e, we can index the element of A as a sequence

$$A = \{x_1, x_2, \ldots, x_n, \ldots\}$$

Discrete random variable

Definition (Probability Distribution of a Discrete Random Variable)

The probability distribution of a discrete random variable is the set of numbers $P(X = x)$ or each value x that X can take. The distribution takes the value 0 at all other numbers. Notice that the distribution is non-negative. The probability distribution is also known as the **probability mass function**.

Represent the probability mass function

- As a table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- As a function:

$$p(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 1, 2, 3, \ldots, \\ 0 & \text{elsewhere} \end{cases}$$

Example

Example

We flip a biased coin two times. The flips are independent.The coin has $P(H) = 0.7$ and $P(T) = 0.3$. We record a 1 when it comes up heads, and when it comes up tails, we record a 0. What is the probability distribution of the sum of the outcomes of the flips?

Joint and Conditional Probability for Random Variables

Joint probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Simplified notations

- We will write $P(X)$ to denote the probability distribution of a random variable
- We will write $P(x)$ or $P(X = x)$ to denote the probability that random variable takes a particular value
- Example

$$P(\{X = x\}|\{Y = y\})P(\{Y = y\}) = P(\{X = x\} \cap \{Y = y\})$$

can be written as

$$P(x|y)P(y) = P(x, y)$$

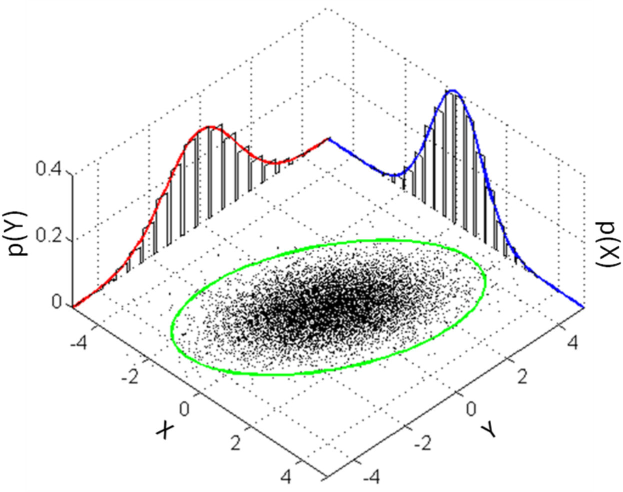
Marginal probability

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Marginal probability



Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Example

Example

Measurements for the length and width of rectangular plastic covers for CDs are rounded to the nearest mm. Let X denote the length and Y denote the width. Assume that the joint probability of X and Y is represented by the following table

		x=length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

What is the probability distribution of X?

Example

Example

Measurements for the length and width of rectangular plastic covers for CDs are rounded to the nearest mm. Let X denote the length and Y denote the width. Assume that the joint probability of X and Y is represented by the following table

		x=length		
		129	130	131
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	16	0.08	0.28	0.04

Are X and Y independent?