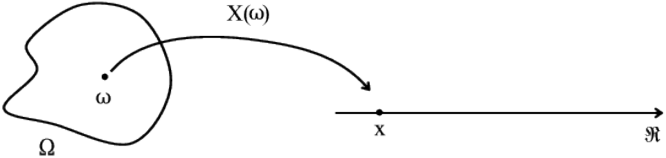
MATH 205: Statistical methods

September 29th, 2021

Lecture 8: Continuous random variables

Random variables and expectations

Random variables



Notations:

- random variables are denoted by uppercase letters (e.g., X);
- the observed values of the random variables are denoted by lowercase letters (e.g., x)

Discrete random variable

Definition

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

Note: A set A is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers, i.e, we can index the element of A as a sequence

$$A = \{x_1, x_2, \dots, x_n, \dots\}$$

Discrete random variable

Definition (Probability Distribution of a Discrete Random Variable)

The probability distribution of a discrete random variable is the set of numbers $P(X = x)$ or each value x that X can take. The distribution takes the value 0 at all other numbers. Notice that the distribution is non-negative. The probability distribution is also known as the **probability mass function**.

Represent the probability mass function

- As a table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- As a function:

$$p(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 1, 2, 3, \dots, \\ 0 & \text{elsewhere} \end{cases}$$

Joint probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Marginal probability

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Example

Example

Measurements for the length and width of rectangular plastic covers for CDs are rounded to the nearest mm. Let X denote the length and Y denote the width. Assume that the joint probability of X and Y is represented by the following table

		x=length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

What is the probability distribution of X?

Example

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		x=length		
		129	130	131
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Are X and Y independent?

Example

Example

Assume that the joint probability of X and Y is represented by the following table

		Y		
		1	**2**	**3**
X	**1**	0.32	0.03	0.01
	2	0.06	0.24	0.02
	3	0.02	0.03	0.27

What are the probability distribution of X and Y? Are they independent?

Continuous random variables

Continuous random variables

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **absolutely continuous** or, for simplicity, **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

Whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies the equation above.

Properties

Let X be a continuous r.v. with density function f, then

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\ dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

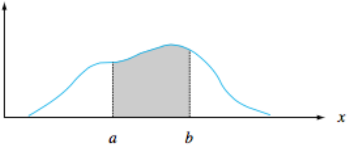

Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(X \in [0.25, 0.75])$

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t)$$

is called the distribution function of X.

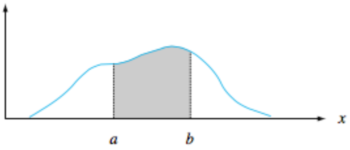

Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Distribution function

For continuous random variable:

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx = F(b) - F(a)$$

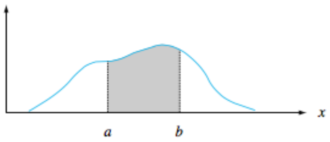

Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Moreover:

$$f(x) = F'(x)$$

Example

Problem

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(y) = \begin{cases} 1 - \frac{16}{y^2} & \text{if } y \geq 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P[4 \leq Y \leq 8]$.