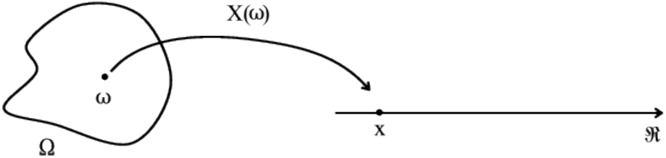
MATH 205: Statistical methods

October 4th, 2021

Lecture 9: Expectations

Random variables and expectations

Random variables



Notations:

- random variables are denoted by uppercase letters (e.g., X);
- the observed values of the random variables are denoted by lowercase letters (e.g., x)

Discrete random variable

Definition

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

Represent the probability mass function

- As a table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

- As a function:

$$p(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 1, 2, 3, \ldots, \\ 0 & \text{elsewhere} \end{cases}$$

Continuous random variables

Definition

Let X be a random variable. Suppose that there exists a nonnegative real-valued function $f : \mathbb{R} \to [0, \infty)$ such that for any subset of real numbers A, we have

$$P(X \in A) = \int_A f(x)dx$$

Then X is called **continuous**. The function f is called the **probability density function**, or simply the **density function** of X.

Properties

Let X be a continuous r.v. with density function f, then

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- For any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

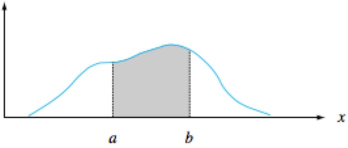


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t) = \int_{(-\infty, t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

is called the distribution function of X.

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx = F(b) - F(a)$$

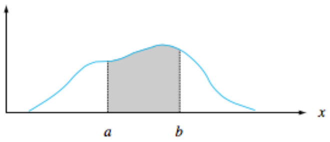


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Moreover:

$$f(x) = F'(x)$$

Joint probability and marginal probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Expectations

Expected values

Assume that we are playing a game.

- Toss a fair coin 2 times
- For every head, I'll give you one dollar. For every tail, I'll give you four dollars.

But you have to pay c dollars to me to play the game.
What is the maximum amount c would you pay to play the game?

Expected values

The distribution of the amount X you're getting out of one game is

X	2	5	8
probability	0.25	0.5	0.25

Question: What is the (theoretical) average of the amount that you would get out of one game?

Expected value: discrete variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability distribution P, we define the expected value of X as

$$\mathbb{E}[X] = \sum_{x \in \mathcal{D}} x P(X = x)$$

This is sometimes written $\mathbb{E}_P[X]$, to clarify which distribution one has in mind.

Expected value: discrete variables

Example

We agree to play the following game. I flip a fair coin. If the coin comes up heads, you pay me 2; if the coin comes up tails, I pay you 1. What is the expected value of my income?

Expected value: discrete variables

Definition

Assume we have a function f that maps a discrete random variable X into a set of numbers D_f. Then f(X) is a discrete random variable, too, which we write F. The expected value of this random variable is written

$$\mathbb{E}[f(X)] = \sum_{u \in \mathcal{D}_f} uP(F = u) = \sum_{x \in \mathcal{D}} f(x)P(X = x)$$

which is sometimes referred to as "the expectation of f". The process of computing an expected value is sometimes referred to as "taking expectations".
This is sometimes written $\mathbb{E}[f]$, or $\mathbb{E}_P[f]$ or $\mathbb{E}_{P(X)}[f]$.

Exercise

Problem

A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

- *What is* $\mathbb{E}[X^2 - X]$*?*
- *Compute* $\mathbb{E}[2^X]$

Expected value: continuous variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability density function $p(x)$, we define the expected value of X as

$$\mathbb{E}[X] = \int_{\mathcal{D}} xp(x)\ dx$$

This is sometimes written $\mathbb{E}_P[X]$, to clarify which distribution one has in mind.

Expected value: continuous variables

Definition

Assume we have a function f that maps a discrete random variable X into a set of numbers D_f. Then f(X) is a continuous random variable, too, which we write F. The expected value of this random variable is written

$$\mathbb{E}[f(X)] = \int_{\mathcal{D}} f(x)p(x)\ dx$$

which is sometimes referred to as "the expectation of f". The process of computing an expected value is sometimes referred to as "taking expectations".
This is sometimes written $\mathbb{E}[f]$, or $\mathbb{E}_P[f]$ or $\mathbb{E}_{P(X)}[f]$.

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[X]$ and $\mathbb{E}(X^2)$.

Mean and variance

Definition

- The mean or expected value of a random variable X is

$$\mathbb{E}[X]$$

- The variance of a random variable X is

$$var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

- The standard deviation of a random variable X is defined as

$$std(X) = \sqrt{var(X)}$$

Exercise

Problem

A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

Compute $var(X)$.

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $var(X)$.