MATH 205: Statistical methods

October 20th, 2021

Lecture 14: Distribution of the sample mean

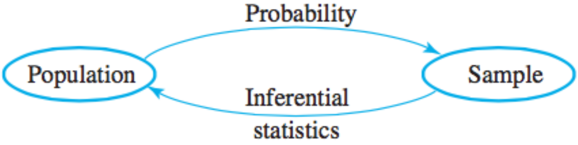
Announcements

- Midterm exam (written part): next Wednesday.
  The exam covers everything up to Chapter 6 (Confidence Interval).
- Midterm exam (simulations):
  - Section 050L: next Monday
  - Section 051L: next Wednesday

  The exam covers everything up Central Limit Theorem.

Chapter 6: Samples and Populations

Random sample



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

The sample mean is an estimate of the population mean

Definition

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution. The sample mean is defined as

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

- The sample mean is a random variable.
- It is random, because different samples from the population will have different values of the sample mean.

Reminder: notations

- Let $X_1, X_2, \ldots, X_n$ be a random sample of size n
- The sample mean of $X_1, X_2, \ldots, X_n$, defined by

$$\bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n},$$

is a random variables

- When the values of $x_1, x_2, \ldots, x_n$ are collected,

$$\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n},$$

is a realization of the $\bar{X}$, and is a number

The sample mean is an estimate of the population mean

Questions:

- What can we say about the distribution of $\bar{X}$?
- When can wee use $\bar{X}$ to estimate the population mean with confidence?

Linear combination of random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables (with possibly different means and/or variances). Define

$$T = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

then the mean and the standard deviation of T can be computed by

- $E(T) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$
- $Var(T) = a_1^2Var(X_1) + a_2^2Var(X_2) + \ldots + a_n^2Var(X_n)$

Linear combination of normal random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent normal random variables (with possibly different means and/or variances). Then

$$T = a_1 X_1 + a_2 X_2 + \ldots a_n X_n$$

also follows the normal distribution with

- $E(T) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$
- $Var(T) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

Mean and variance of the sample mean

Theorem

Given independent random samples $X_1, X_2, ..., X_n$ *from a distribution with mean* μ *and standard deviation* σ*, the mean is modeled by a random variable* $\bar{X}$*,*

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Then

$$E[\bar{X}] = \mu$$

and

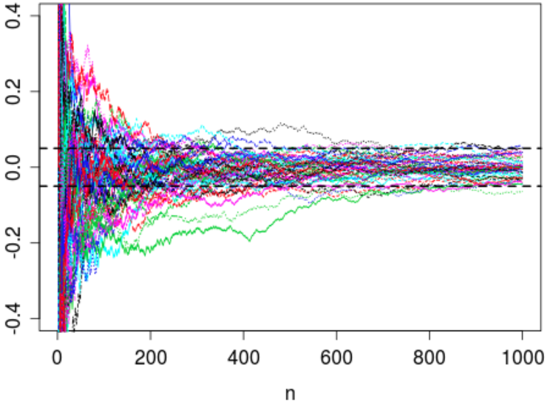
$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

Law of large numbers

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then

$$\bar{X} \to \mu$$

as n approaches infinity



The Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then, in the limit when $n \to \infty$, the standardized version of $\bar{X}$ have the standard normal distribution

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z\right) = \mathbb{P}[Z \le z] = \Phi(z)$$

Rule of Thumb:

If $n > 30$, the Central Limit Theorem can be used for computation.

Example

Problem

Let $X_1, X_2, \ldots, X_{64}$ *be a random sample from a distribution with population mean* $\mu = 1$ *and standard deviation* $\sigma = 2$*.*
Let $\bar{X}$ *be the sample mean*

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_{64}}{64}$$

Compute $P[\bar{X} \leq 1.49]$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Example

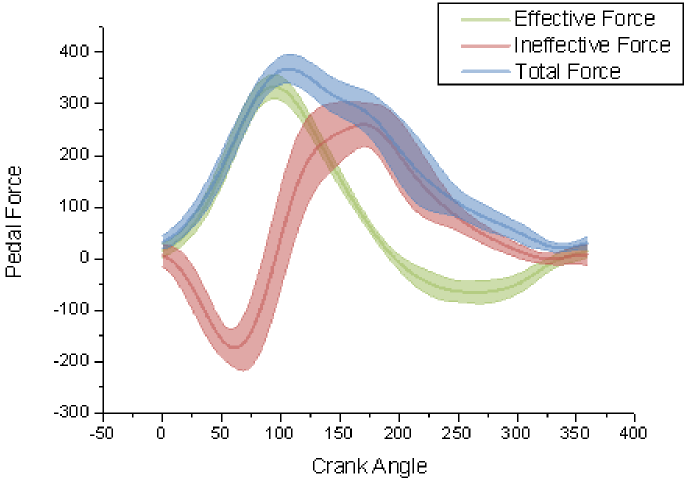
Problem

When a batch of a certain chemical product is prepared, the amount of a particular impurity in the batch is a random variable with mean value 4.0 g and standard deviation 1.5 g.

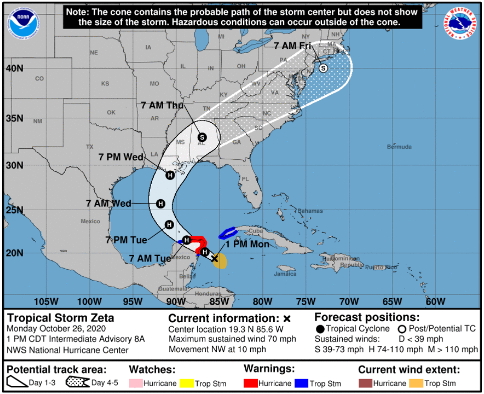
If 50 batches are independently prepared, what is the (approximate) probability that the sample average amount of impurity is between 3.5 and 3.8 g?

Confidence Intervals

A good prediction comes with a range



A good prediction comes with a range



A 70% confidence region of the path of a hurricane.

Confidence

- Assume that you have been using an AI to predict the stock price of Microsoft every day in the last few years
- The prediction comes as a range, e.g., [295, 305]
- The algorithm, on average, is correct 95 out of 100 days
- Then we say that a prediction from this AI has a confidence of 95%

Confidence interval: Example 1

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ *(unknown) and standard deviation* $\sigma = 0.85$*. A random sample of* $n = 25$ *specimens is selected with sample average* $\bar{X}$*.*
Find a number c such that

$$P\left[-c < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < c\right] = 0.95$$

Confidence interval

- We have

$$P\left[-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right] = 0.95$$

- Rearranging the inequalities gave

$$P\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

- This means that if you use

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ, then you are correct 95% of the time.

Normal distribution with know σ

- Using

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ is correct 95% of the time.

- If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

z-critical value

NOTATION z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

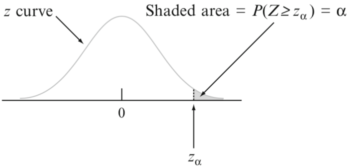


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1-\alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as ***z* critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

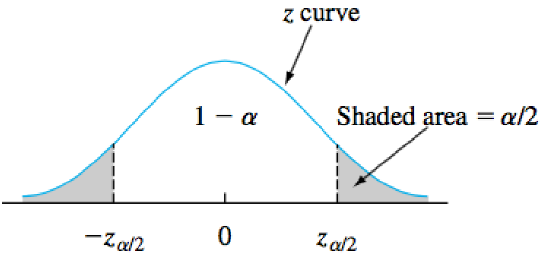


Figure 8.4 $P(-z_{\alpha/2} \le Z \le z_{\alpha/2}) = 1 - \alpha$

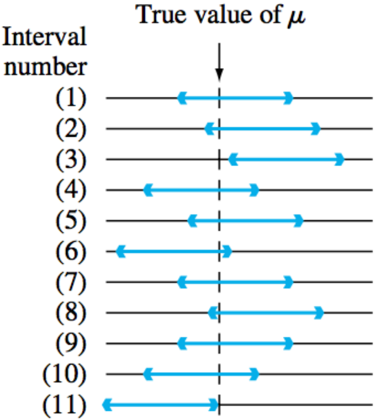
$100(1-\alpha)\%$ confidence interval

A $100(1-\alpha)\%$ confidence interval for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Interpreting confidence intervals

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT correct.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is good.
- Saying "if we repeat the experiment many times, the interval contains μ about 95% of the time" is perfect.

Example

Example

Assume that the helium porosity (in percentage) of coal samples taken from any particular seam is normally distributed with true standard deviation $\sigma = .75$.

- Compute a 95% CI for the true average porosity of a certain seam if the average porosity for 20 specimens from the seam was 4.85.
- How large a sample size is necessary if the width of the 95% interval is to be .40?

One-sided CIs (Confidence bounds)

Example 1b: One-sided confidence interval

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0.85$. A random sample of $n = 25$ specimens is selected with sample average $\bar{X}$.
Find a number b such that

$$P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < b\right] = 0.95$$

CIs vs. one-sided CIs

CIs:

- $100(1-\alpha)\%$ confidence

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

One-sided CIs:

- $100(1-\alpha)\%$ confidence

$$\left(-\infty, \bar{x} + z_{\alpha}\frac{s}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(-\infty, \bar{x} + 1.64\frac{s}{\sqrt{n}}\right)$$