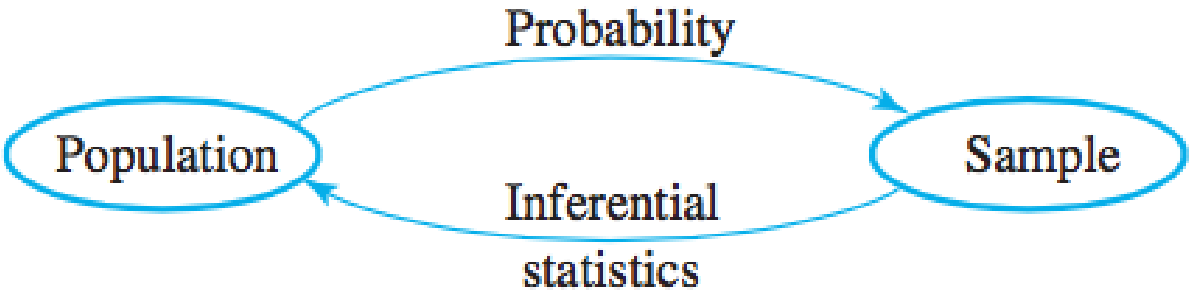
MATH 205: Statistical methods

November 1st, 2021

Lecture 16: Confidence intervals

Chapter 6: Samples and Populations

Random sample



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

The sample mean is an estimate of the population mean

Definition

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution. The sample mean is defined as

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

- The sample mean is a random variable.
- It is random, because different samples from the population will have different values of the sample mean.
- $\bar{X}$ vs. $\bar{x}$

The sample mean is an estimate of the population mean (the expected value)

Questions:

- What can we say about the distribution of $\bar{X}$?
- When can we use $\bar{X}$ to estimate the population mean with confidence?

Linear combination of random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables (with possibly different means and/or variances). Define

$$T = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

then the mean and the standard deviation of T can be computed by

- $E(T) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$
- $Var(T) = a_1^2Var(X_1) + a_2^2Var(X_2) + \ldots + a_n^2Var(X_n)$

Linear combination of normal random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent normal random variables (with possibly different means and/or variances). Then

$$T = a_1 X_1 + a_2 X_2 + \ldots a_n X_n$$

also follows the normal distribution with

- $E(T) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$
- $Var(T) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$

Mean and variance of the sample mean

Theorem

Given independent random samples $X_1, X_2, ..., X_n$ from a distribution with mean μ and standard deviation σ, the mean is modeled by a random variable $\bar{X}$,

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Then

$$E[\bar{X}] = \mu$$

and

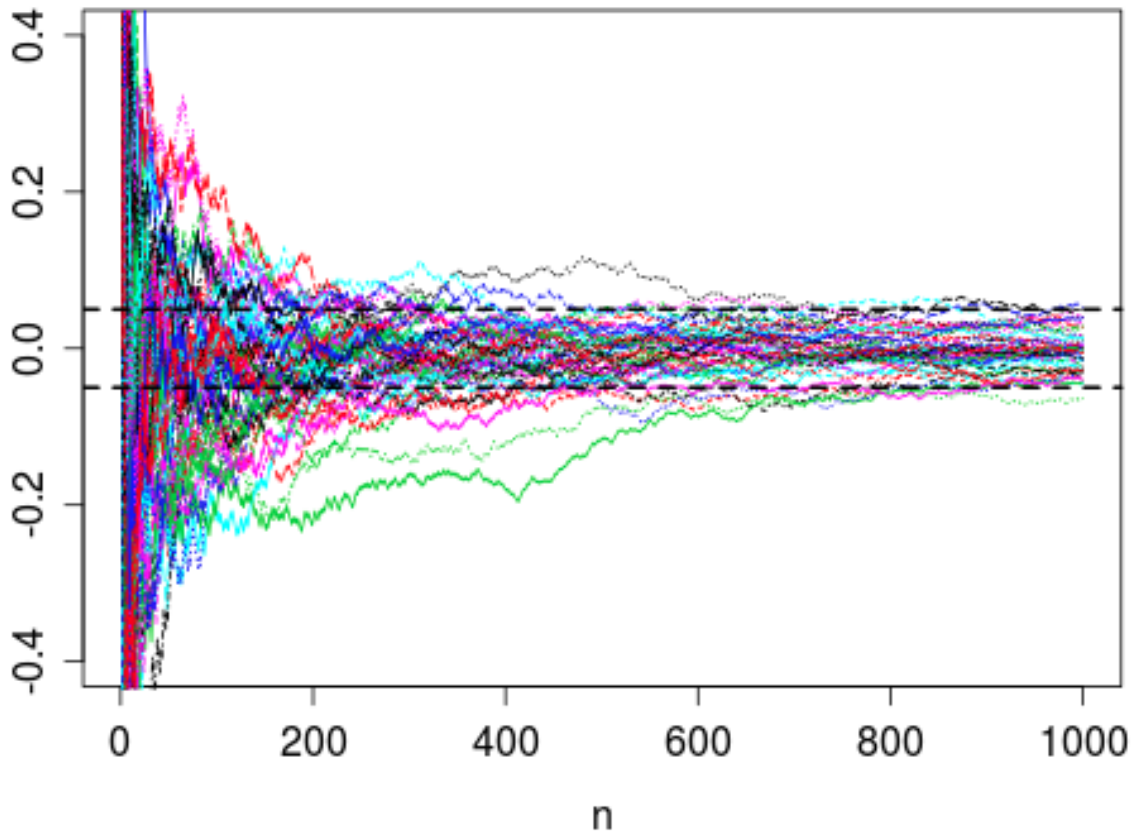
$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

Law of large numbers

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then

$$\bar{X} \to \mu$$

as n approaches infinity



The Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then, in the limit when $n \to \infty$, the standardized version of $\bar{X}$ have the standard normal distribution

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z\right) = \mathbb{P}[Z \leq z] = \Phi(z)$$

Rule of Thumb:

If $n > 30$, the Central Limit Theorem can be used for computation.

Example

Problem

Let $X_1, X_2, \ldots, X_{64}$ be a random sample from a distribution with population mean $\mu = 1$ and standard deviation $\sigma = 2$.
Let $\bar{X}$ be the sample mean

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_{64}}{64}$$

Compute $P[\bar{X} \leq 1.49]$

Example

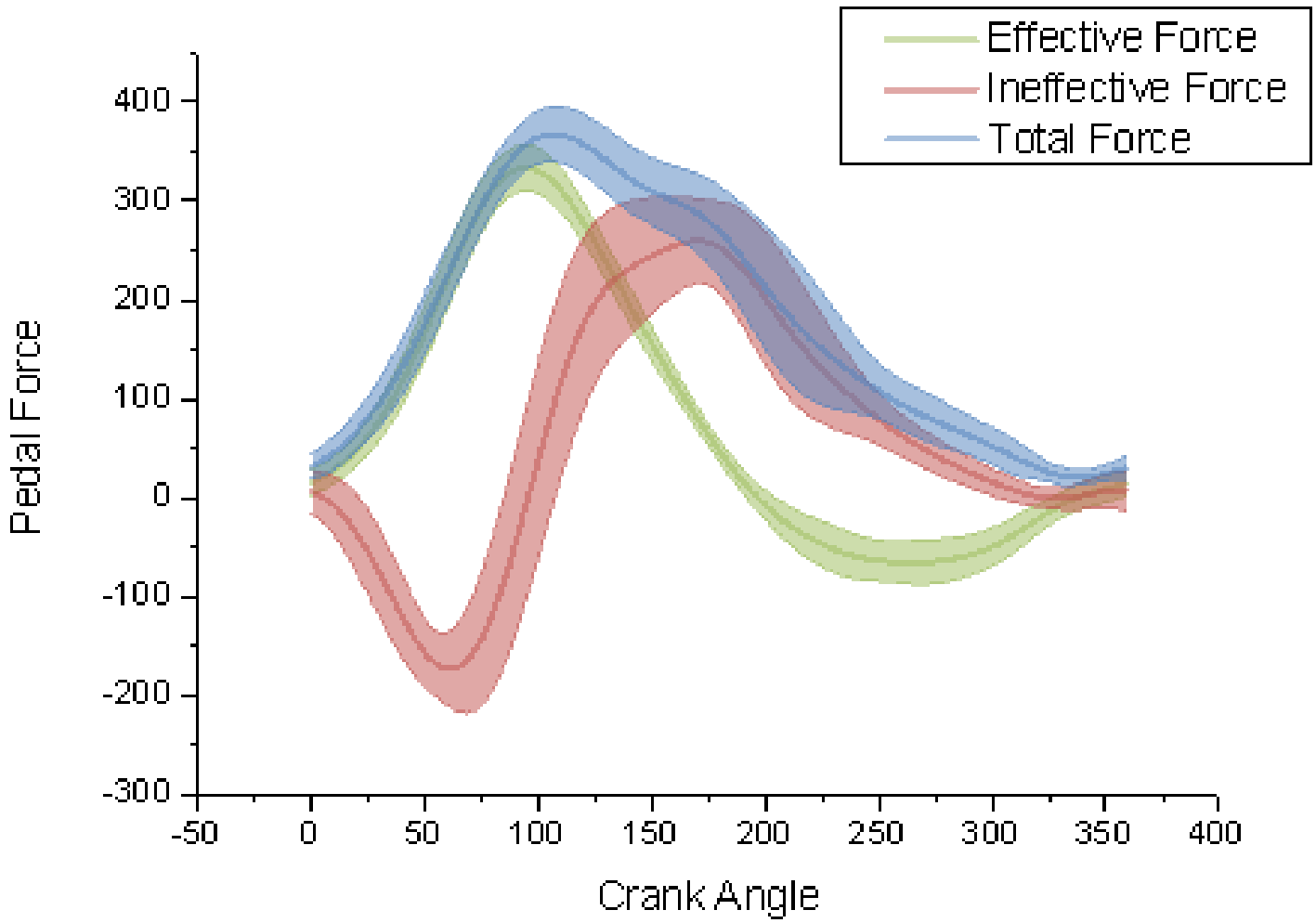
Problem

When a batch of a certain chemical product is prepared, the amount of a particular impurity in the batch is a random variable with mean value 4.0 g and standard deviation 1.5 g.

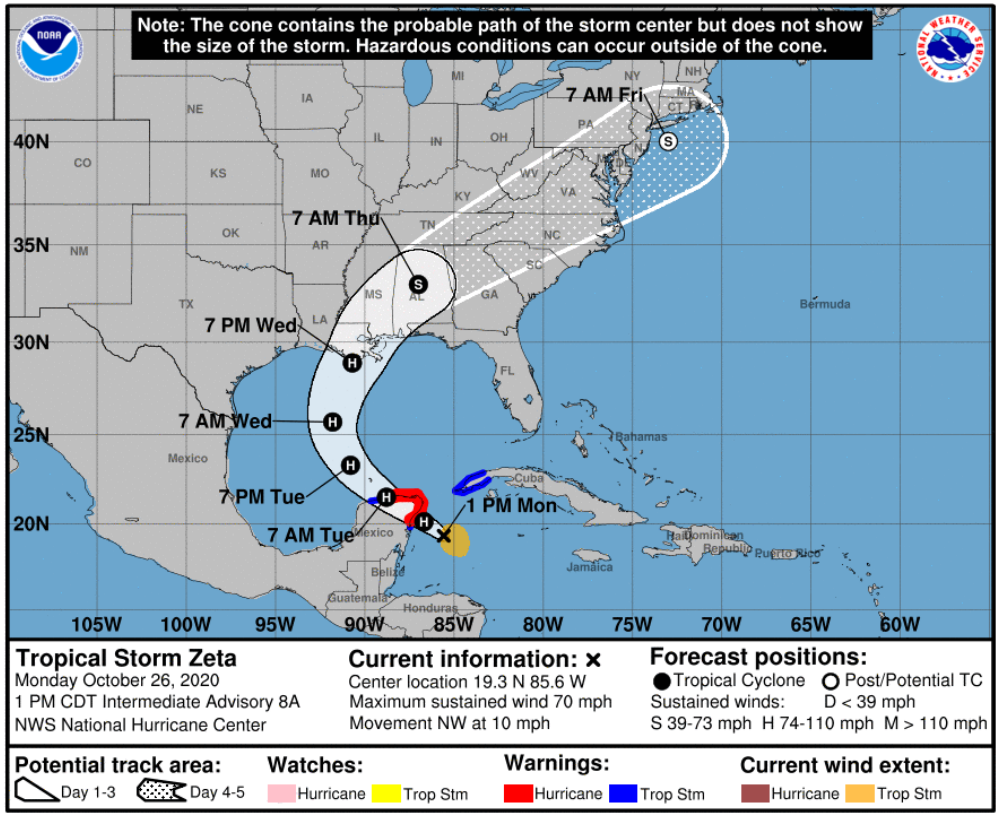
If 50 batches are independently prepared, what is the (approximate) probability that the sample average amount of impurity is between 3.5 and 3.8 g?

Confidence Intervals

A good prediction comes with a range



A good prediction comes with a range



A 70% confidence region of the path of a hurricane.

Confidence

- Assume that you have been using an AI to predict the stock price of Microsoft every day in the last few years
- The prediction comes as a range, e.g., [295, 305]
- The algorithm, on average, is correct 95 out of 100 days
- Then we say that a prediction from this AI has a confidence of 95%

Confidence interval: Example 1

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0.85$. A random sample of $n = 25$ specimens is selected with sample average $\bar{X}$.
Find a number c such that

$$P\left[-c < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < c\right] = 0.95$$

Confidence interval

- We have

$$P\left[-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right] = 0.95$$

- Rearranging the inequalities gave

$$P\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

- This means that if you use

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ, then you are correct 95% of the time.

Assumption: Normal distribution with known σ

- Using
$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$
as a range to estimate μ is correct 95% of the time.
- If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then
$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$
is a 95% confidence interval of μ

z-critical value

NOTATION

z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

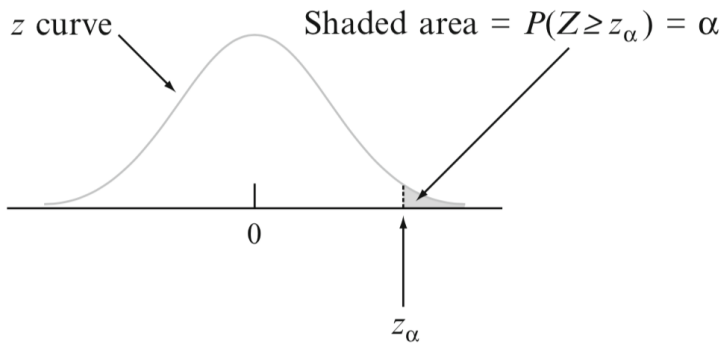


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1 - \alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1 - \alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as **z critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

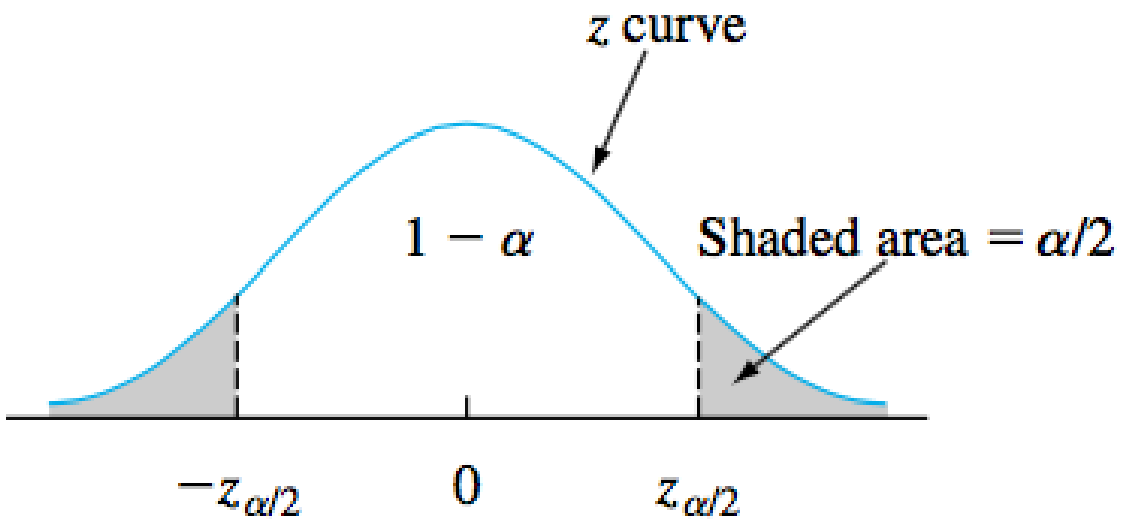


Figure 8.4 $P(-z_{\alpha/2} \le Z \le z_{\alpha/2}) = 1 - \alpha$

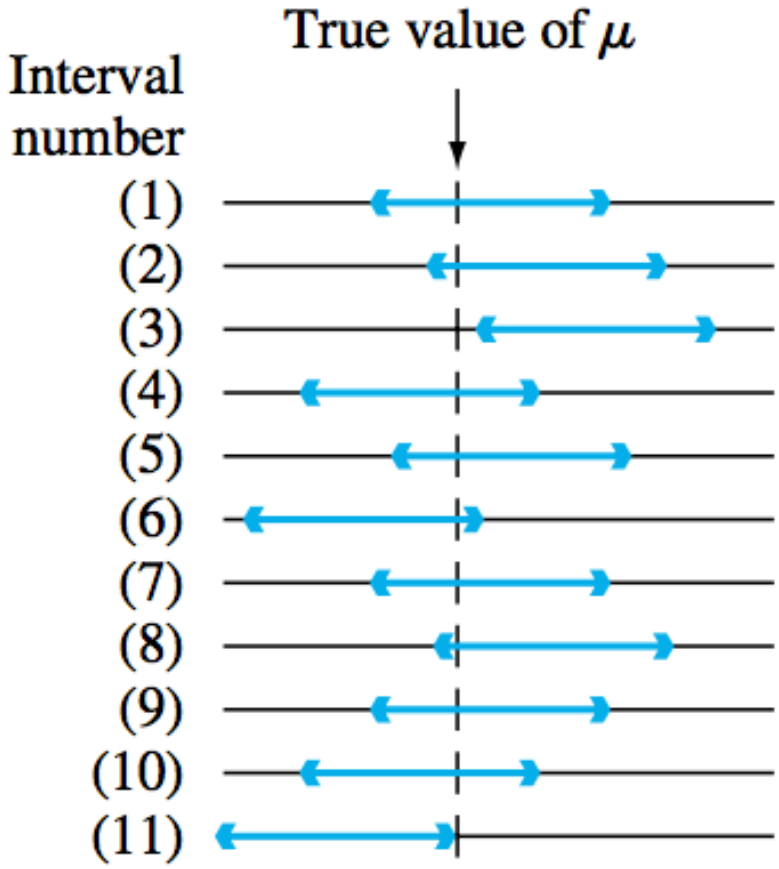
$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Interpreting confidence intervals

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT correct.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is good.
- Saying "if we repeat the experiment many times, the interval contains μ about 95% of the time" is perfect.

Example

Example

Assume that the helium porosity (in percentage) of coal samples taken from any particular seam is normally distributed with true standard deviation $\sigma = .75$.

- Compute a 95% CI for the true average porosity of a certain seam if the average porosity for 20 specimens from the seam was 4.85.
- How large a sample size is necessary if the width of the 95% interval is to be .40?

One-sided CIs (Confidence bounds)

Example 1b: One-sided confidence interval

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0.85$. A random sample of $n = 25$ specimens is selected with sample average $\bar{X}$.
Find a number b such that

$$P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < b\right] = 0.95$$

CIs vs. one-sided CIs

CIs:

- $100(1-\alpha)\%$ confidence

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

One-sided CIs:

- $100(1-\alpha)\%$ confidence

$$\left(-\infty, \bar{x} + z_{\alpha}\frac{s}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(-\infty, \bar{x} + 1.64\frac{s}{\sqrt{n}}\right)$$

Confidence level

Problem

Determine the confidence level for each of the following large-sample confidence intervals/bounds:

(a) $\bar{x} + 0.84s/\sqrt{n}$

(b) $\left(\bar{x} - 0.84s/\sqrt{n}, \bar{x} + 0.84s/\sqrt{n}\right)$

(c) $\bar{x} - 2.05s/\sqrt{n}$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Example

Example

A sample of 66 obese adults was put on a low-carbohydrate diet for a year. The average weight loss was 11 lb and the standard deviation was 19 lb. Calculate a 99% lower confidence bound for the true average weight loss

Assumptions

- So far
  - Normal distribution
  - σ is known
- Large-sample setting
  - ~~Normal distribution~~
    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$

Measures of Variability: deviations from the mean

Given a data set $x_1, x_2, \ldots, x_n$, the sample standard deviation, denoted by s, is given by

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

Principles

- Central Limit Theorem

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

  is approximately normal when $n > 30$
- Moreover, when n is sufficiently large $s \approx \sigma$
- Conclusion:

$$\frac{\bar{X} - \mu}{s/\sqrt{n}}$$

  is approximately normal when n is sufficiently large

If $n > 40$, we can ignore the normal assumption and replace σ by s

95% confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2$,. . . , $X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2$,. . . , $X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ