MATH 205: Statistical methods

November 3rd, 2021

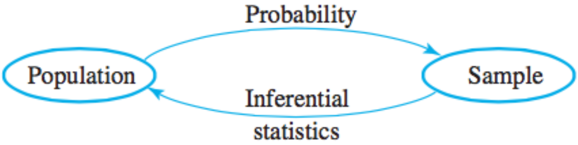
Lecture 17: Confidence intervals (cont)

Announcements

- Homework due next Wednesday
- Quiz next Monday (Confidence intervals)

Chapter 6: Samples and Populations

Random sample



Definition

The random variables $X_1, X_2, ..., X_n$ are said to form a (simple) random sample of size n if

1. the X_i's are independent random variables
2. every X_i has the same probability distribution

Mean and variance of the sample mean

Theorem

Given independent random samples $X_1, X_2, ..., X_n$ from a distribution with mean μ and standard deviation σ, the mean is modeled by a random variable $\bar{X}$,

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Then

$$E[\bar{X}] = \mu$$

and

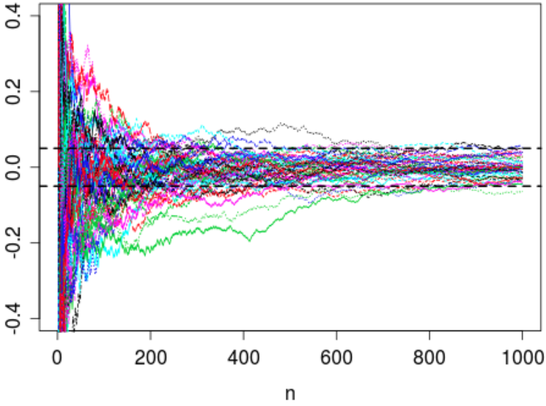
$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

Law of large numbers

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then

$$\bar{X} \to \mu$$

as n approaches infinity



The Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then, in the limit when $n \to \infty$, the standardized version of $\bar{X}$ have the standard normal distribution

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z\right) = \mathbb{P}[Z \leq z] = \Phi(z)$$

Rule of Thumb:

If $n > 30$, the Central Limit Theorem can be used for computation.

Assumption: Normal distribution with known σ

- Using

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ is correct 95% of the time.

- If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

z-critical value

NOTATION z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

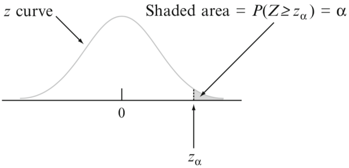


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1 - \alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as ***z* critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

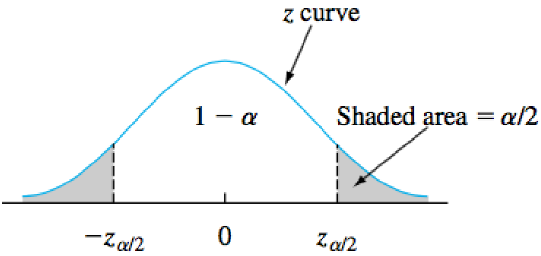


Figure 8.4 $P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha$

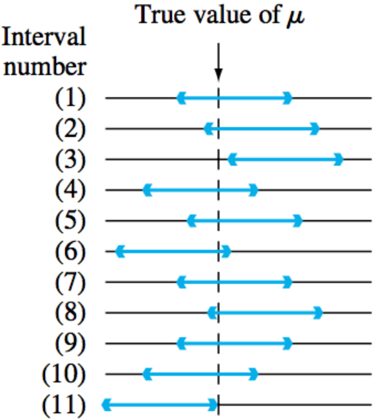
$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Probability $\to$ Confidence

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT correct.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is good.

CIs vs. one-sided CIs

CIs:

- $100(1-\alpha)\%$ confidence

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

One-sided CIs:

- $100(1-\alpha)\%$ confidence

$$\left(-\infty, \bar{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(-\infty, \bar{x} + 1.64\frac{\sigma}{\sqrt{n}}\right)$$

Confidence level

Problem

Determine the confidence level for each of the following large-sample confidence intervals/bounds:

(a) $\bar{x} + 0.84\sigma/\sqrt{n}$

(b) $\left(\bar{x} - 0.84\sigma/\sqrt{n}, \bar{x} + 0.84\sigma/\sqrt{n}\right)$

(c) $\bar{x} - 2.05\sigma/\sqrt{n}$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Assumptions

- So far
  - Normal distribution
  - σ is known
- Large-sample setting
  - ~~Normal distribution~~
    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$

Measures of Variability: deviations from the mean

Given a data set $x_1, x_2, \ldots, x_n$, the sample standard deviation, denoted by s, is given by

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}}$$

Principles

- Central Limit Theorem
$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$
is approximately normal when $n > 30$
- Moreover, when n is sufficiently large $s \approx \sigma$
- Conclusion:
$$\frac{\bar{X} - \mu}{s/\sqrt{n}}$$
is approximately normal when n is sufficiently large

If $n > 40$, we can ignore the normal assumption and replace σ by s

95% confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$ ($n > 40$), we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

One-sided CIs

A **large-sample upper confidence bound for μ** is

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

and a **large-sample lower confidence bound for μ** is

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}$$

Example

Example

A random sample of 50 patients who had been seen at an outpatient clinic was selected, and the waiting time to see a physician was determined for each one, resulting in a sample mean time of 40.3 min and a sample standard deviation of 28.0 min.

- Construct an 95% confidence interval of the true average waiting time.
- Assuming it is known that the true standard deviation of the waiting time is 27 min, construct an 95% confidence interval of the true average waiting time.