MATH 205: Statistical methods

Lecture 8: Basic properties of probability

Announcements

- Homework 1 due this Friday, before lecture
- Quiz 1 this Wednesday (next lecture). Cover materials of Lecture 1–6.

Topics

- Sample space and events
- Basic properties of probability
- Advanced properties of probability
- Compute probability
  - Computing event probabilities by counting outcomes
  - Computing probabilities by reasoning about sets

Sample space and events

1. An experiment: is any action, process, or phenomenon whose outcome is subject to uncertainty
2. An outcome: is a result of an experiment
   Each run of the experiment results in one outcome
3. A sample space: is the set of all possible outcomes of an experiment
4. An event: is a subset of the sample space.
   An event occurs when one of the outcomes that belong to it occurs

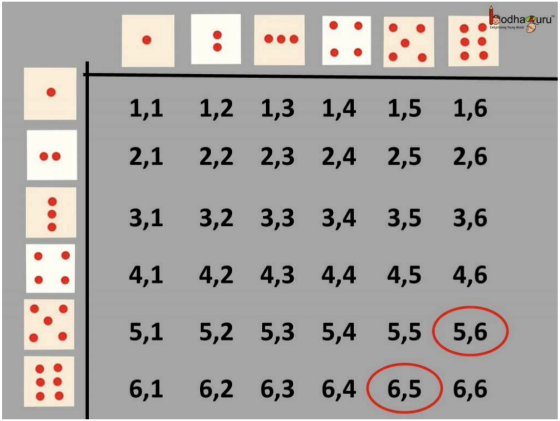
Sample space and events: example

1. Experiment: Toss a regular dice
2. Sample space: $\{1, 2, 3, 4, 5, 6\}$
3. Some events
   - E_1 = the result is an even number = $\{2, 4, 6\}$
   - E_2 = the result is greater than 2= $\{3, 4, 5, 6\}$
   - $E_3 = \{1, 3, 5, 6\}$

Define probability

1. Experiment: Toss two regular dice
2. E_1 = the summation of the two dice is 11

$$P[E_1] = 1/18$$



1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Different probabilities

- Given an experiment and a sample space, we can define many different probabilities
- Experiment: tossing a coin, $\Omega = \{H, T\}$
- If you believe the coin is fair:

$$P(\emptyset) = 0, \quad P(\{H\}) = 0.5, \quad P(\{T\}) = 0.5, \quad P(\{H, T\}) = 1.$$

- If you do not, then maybe

$$P(\emptyset) = 0, \quad P(\{H\}) = 0.7, \quad P(\{T\}) = 0.3, \quad P(\{H, T\}) = 1.$$

Different views of probability

- Frequentist: The probability of an outcome is the frequency of that outcome in a very large number of repeated experiments
- Bayesian: Probability is a quantification of a belief about how often an outcomes occurs

Some fundamental questions

Probability is a function defined on the set of events of an experiment

1. What conditions should we impose to define probability?
2. How should we adapt probability model with new information?

Basic set operators

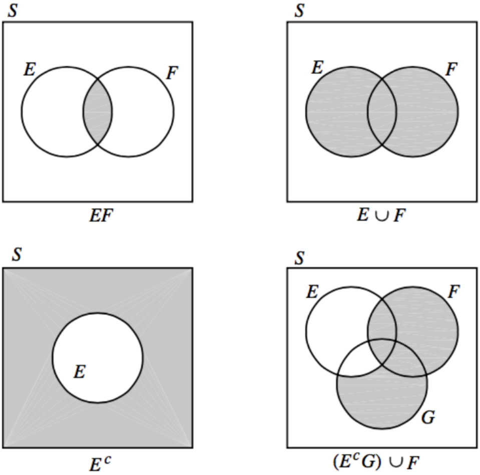


Figure 1.1 Venn diagrams of the events specified.

What conditions should we impose to define probability?

- P[the sample space] $= 1$
- $0 \leq P[E] \leq 1$ for all events E
- $P[\emptyset] = 0$
- If E_1 and E_2 are disjoint then $P[E_1 \cup E_2] = P[E_1] + P[E_2]$
- If E_1, E_2 and E_3 are mutually disjoint then

$$P[E_1 \cup E_2 \cup E_3] = P[E_1] + P[E_2] + P[E_3]$$

Basic properties of probability

Useful Facts 3.1 (Basic Properties of the Probability Events)
We have

- The probability of every event is between zero and one; in equations

$$0 \le P(\mathcal{A}) \le 1$$

  for any event $\mathcal{A}$.
- Every experiment has an outcome; in equations,

$$P(\Omega) = 1.$$

- The probability of disjoint events is additive; writing this in equations requires some notation. Assume that we have a collection of events $\mathcal{A}_i$, indexed by i. We require that these have the property $\mathcal{A}_i \cap \mathcal{A}_j = \varnothing$ when $i \neq j$. This means that there is no outcome that appears in more than one $\mathcal{A}_i$. In turn, if we interpret probability as relative frequency, we must have that

$$P(\cup_i \mathcal{A}_i) = \sum_i P(\mathcal{A}_i)$$

Advanced properties of probability

Useful Facts 3.2 (Properties of the Probability of Events)

- $P(\mathcal{A}^c) = 1 - P(\mathcal{A})$
- $P(\varnothing) = 0$
- $P(\mathcal{A} - \mathcal{B}) = P(\mathcal{A}) - P(\mathcal{A} \cap \mathcal{B})$
- $P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A} \cap \mathcal{B})$

Others

Problem

If $A \subset B$*, then* $P(A) \leq P(B)$*.*

Others

Problem

For any events A, B

$$P(A) = P(A \cap B) + P(A \cap B^c)$$