MATH 205: Statistical methods

Lecture 12: Conditional probability

Tentative schedule

Date	Theme/Topic	Labs	Assignments
Aug 31	Syllabus		
Sep 2—9	Chapter 1: Describing dataset	Section 2: Handling data	
Sep 12—16	Chapter 2: Looking at Relationships	Section 3: Univariate data	
Sep 19—23	Chapter 3: Basic Ideas in Probability	Section 4: Bivariate Data	Homework 1 (due 09/23)
Sep 26—30	Chapters 3-4	Section 4: Correlation	
Oct 3—7	Chapter 4: Random variables and expectations	Section 6: Random data	Homework 2 (due 10/07)
Oct 10—14	Chapter 5: Useful distributions	Section 7: The central limit theorem	
Oct 17—21	Chapter 6: Samples and populations	Section 9: Confidence interval estimation	Homework 3 (due 10/21)
Oct 24—28	Review Midterm exam		Midterm: Oct 28 (lecture) Oct 24-26 (labs)
Oct 31—Nov 4	Chapter 7: The significance of evidence	Section 10: Hypothesis testing	
Nov 7—11	Goodness of Fit	Section 12: Goodness of Fit	Homework 4 (due 11/11)
Nov 14—18	Linear Regression	Section 13: Linear regression	
Nov 21–25	Thanksgiving break		
Nov 28 —Dec 2	One-Way Analysis of Variance	Section 15: Analysis of variance	Homework 5 (due 12/02)
Dec 5—7	Selected topics + Review		
Exam week			

Chapter 3: Basic ideas in probability

- Experiments, outcomes, events, and probability
- Independence
- Conditional probability

Example: the JEDI contract (2019)

- is a large United States Department of Defense cloud computing contract that worths 10 billion.
- three outcomes: All-others (1), Microsoft (2), and Amazon (3)
$$\Omega = \{1, 2, 3\}$$
- Let's say, originally, we believed that
$$P(1) = 1/5, \quad P(2) = 2/5, \quad P(3) = 2/5$$

Example: the JEDI contract

Amazon Accuses Trump of 'Improper Pressure' on JEDI Contract

In a legal complaint, Amazon said the president had attacked it behind the scenes to harm its C.E.O., Jeff Bezos, "his perceived political enemy."

Amazon had been considered the front-runner for the Joint Enterprise Defense Infrastructure project, known as JEDI. Mark Lennihan/Associated Press

Probability updated with new information

- Let's say, originally, we believe that

$$P(1) = 1/5, \quad P(2) = 2/5, \quad P(3) = 2/5$$

- Suppose that we learn that the outcome is 1 or 2 (This means, the event $A = \{1, 2\}$ happens)
- How should we adapt our model?

Conditional probability

- Denote the new probability by $\tilde{P}$
- We know $\tilde{P}(3) = 0$, and $\tilde{P}(1) + \tilde{P}(2) = 1$
- The new information should not alter the relative chances of 1 and 2
- We can obtain these by setting

$$\tilde{P}(1) = \frac{P(1)}{P(A)}, \quad \tilde{P}(2) = \frac{P(2)}{P(A)}$$

Conditional probability

Definition

Let $P(A) > 0$, the conditional probability of B given A, denoted by $P(B|A)$, is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Conditional probability

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- This definition does not make sense if $P(A) = 0$ (we will learn how to deal with this later)
- The newly defined probability satisfies the 3 rules of probability

Properties

- Rearrange the definition

$$P(B \cap A) = P(B|A)P(A)$$

$\rightarrow$ sometimes called the **law of multiplication**.

- **Bayes' rule**

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Remember $P(A) = P(A \cap B) + P(A \cap B^c)$? We deduce that

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$\rightarrow$ sometimes called the **law of total probability**.

Law of multiplication

Useful Facts 3.3 (Conditional Probability Formulas)
You should remember the following formulas:

- $P(\mathcal{B}|\mathcal{A}) = \dfrac{P(\mathcal{A}|\mathcal{B})P(\mathcal{B})}{P(\mathcal{A})}$
- $P(\mathcal{A}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c)$
- Assume (a) $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ for $i \neq j$ and (b) $\mathcal{A} \cap (\cup_i \mathcal{B}_i) = \mathcal{A}$; then $P(\mathcal{A}) = \sum_i P(\mathcal{A}|\mathcal{B}_i)P(\mathcal{B}_i)$

Example 1

Problem

We throw two fair six-sided dice. What is the conditional probability that the sum of spots on both dice is greater than six, conditioned on the event that the first die comes up five?

Example 2

Problem

Suppose an urn contains 8 red and 4 white balls. Draw two balls without replacement. What is the probability that both are red?