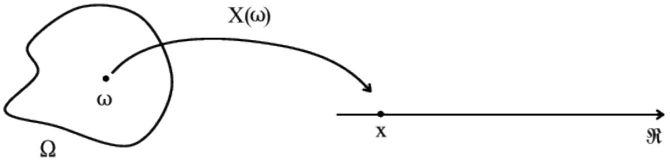
MATH 205: Statistical methods

Lecture 17: Expectation

Random variables



Notations:

- random variables are denoted by uppercase letters (e.g., X);
- the observed values of the random variables are denoted by lowercase letters (e.g., x)

Discrete random variable

A random variables X is discrete if the set of all possible values of X

- is finite
- is countably infinite

A random variables is characterized by its probability mass function, usually represented as a table

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

Continuous random variables

Let X be a continuous r.v. with density function f, then for any fixed constant a, b,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

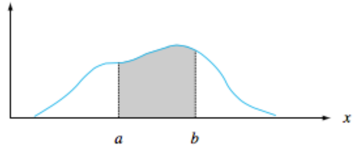


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

Distribution function

Definition

If X is a random variable, then the function F defined on $(-\infty, \infty)$ by

$$F(t) = P(X \leq t) = \int_{(-\infty,t]} f(x)dx$$
$$= \int_{-\infty}^{t} f(x)dx$$

is called the distribution function of X.

Distribution function

For continuous random variable:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

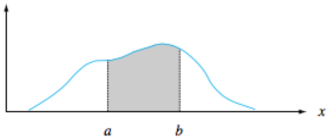
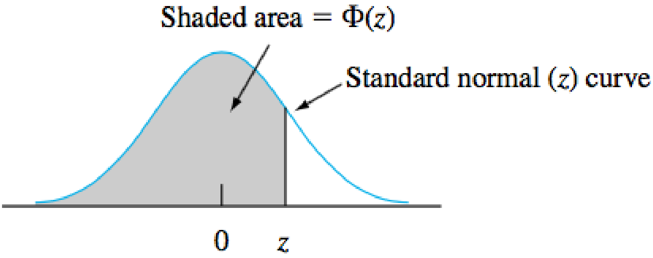


Figure 4.2 $P(a \leq X \leq b)$ = the area under the density curve between a and b

$\Phi(z)$: Distribution function of standard normal



$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(y)\, dy$$

$\Phi(z)$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Expectations

Expected values

Assume that we are playing a game.

- Toss a fair coin 2 times
- For every head, I'll give you one dollar. For every tail, I'll give you four dollars.

The distribution of the amount X you're getting out of one game is

X	2	5	8
probability	0.25	0.5	0.25

Question: What is the (theoretical) average of the amount that you would get out of one game?

Expected value: discrete variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability distribution P, we define the expected value of X as

$$\mathbb{E}[X] = \sum_{x \in \mathcal{D}} x P(X = x)$$

This is sometimes written $\mathbb{E}_P[X]$, to clarify which distribution one has in mind.

Expected value: discrete variables

Definition

Assume we have a function f that maps a discrete random variable X into a set of numbers D_f. Then f(X) is a discrete random variable, too, which we write F. The expected value of this random variable is written

$$\mathbb{E}[f(X)] = \sum_{u \in \mathcal{D}_f} uP(F = u) = \sum_{x \in \mathcal{D}} f(x)P(X = x)$$

which is sometimes referred to as "the expectation of f". The process of computing an expected value is sometimes referred to as "taking expectations".

This is sometimes written $\mathbb{E}[f]$, or $\mathbb{E}_P[f]$ or $\mathbb{E}_{P(X)}[f]$.

Exercise

Problem

A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

- *What is* $\mathbb{E}[X^2 - X]$*?*
- *Compute* $\mathbb{E}[2^X]$

Expected value: continuous variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability density function $p(x)$, we define the expected value of X as

$$\mathbb{E}[X] = \int_{\mathcal{D}} xp(x)\ dx$$

This is sometimes written $\mathbb{E}_P[X]$, to clarify which distribution one has in mind.

Expected value: continuous variables

Definition

Assume we have a function f that maps a discrete random variable X into a set of numbers D_f. Then f(X) is a continuous random variable, too, which we write F. The expected value of this random variable is written

$$\mathbb{E}[f(X)] = \int_{\mathcal{D}} f(x)p(x)\ dx$$

which is sometimes referred to as "the expectation of f". The process of computing an expected value is sometimes referred to as "taking expectations".
This is sometimes written $\mathbb{E}[f]$, or $\mathbb{E}_P[f]$ or $\mathbb{E}_{P(X)}[f]$.

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[X]$ *and* $\mathbb{E}(X^2)$.

Expectations are linear

Useful Facts 4.2 (Expectations Are Linear)
Write f, g for functions of random variables.

- $\mathbb{E}[0] = 0$
- for any constant k, $\mathbb{E}[kf] = k\mathbb{E}[f]$
- $\mathbb{E}[f + g] = \mathbb{E}[f] + \mathbb{E}[g]$.

Mean and variance

Definition

- The mean or expected value of a random variable X is

$$\mathbb{E}[X]$$

- The variance of a random variable X is

$$var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

- The standard deviation of a random variable X is defined as

$$std(X) = \sqrt{var(X)}$$

Example

Problem

Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute $var(X)$.