MATH 205: Statistical methods

Lecture 18: Multivariate distributions

Expectations

Expected value: discrete variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability distribution P:

- The expected value of X is defined as

$$\mathbb{E}[X] = \sum_{x \in \mathcal{D}} x \; P(X = x)$$

- $f(X)$ is a also discrete random variable, and

$$\mathbb{E}[f(X)] = \sum_{x \in \mathcal{D}} f(x) P(X = x)$$

Expected value: continuous variables

Definition

Given a discrete random variable X which takes values in the set $\mathcal{D}$ and which has probability density function $p(x)$:

- The expected value of X is defined as

$$\mathbb{E}[X] = \int_{\mathcal{D}} x p(x) \; dx$$

- $f(X)$ is also a continuous random variable,

$$\mathbb{E}[f(X)] = \int_{\mathcal{D}} f(x) p(x) \; dx$$

Expectations are linear

Useful Facts 4.2 (Expectations Are Linear)
Write f, g for functions of random variables.

- $\mathbb{E}[0] = 0$
- for any constant k, $\mathbb{E}[kf] = k\mathbb{E}[f]$
- $\mathbb{E}[f + g] = \mathbb{E}[f] + \mathbb{E}[g]$.

Mean and variance

Definition

- The mean or expected value of a random variable X is

$$\mathbb{E}[X]$$

- The variance of a random variable X is

$$var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

- The standard deviation of a random variable X is defined as

$$std(X) = \sqrt{var(X)}$$

Properties of variance

We have

- For any constant k, $var[k] = 0$;
- $var[X] \geq 0$;
- $var[kX] = k^2 var[X]$

The variance of a random variable X can be computed by

$$var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Exercise

Problem

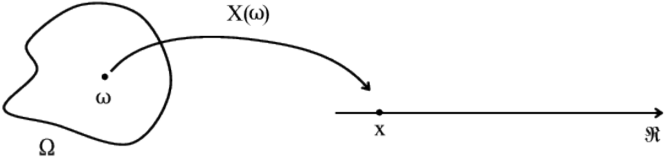
A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

Compute $var(X)$.

Multivariate distributions

Random variables



Notations:

- random variables are denoted by uppercase letters (e.g., X);
- the observed values of the random variables are denoted by lowercase letters (e.g., x)

Joint probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Simplified notations

- We will write $P(X)$ to denote the probability distribution of a random variable
- We will write $P(x)$ or $P(X = x)$ to denote the probability that random variable takes a particular value
- Example

$$P(\{X = x\}|\{Y = y\})P(\{Y = y\}) = P(\{X = x\} \cap \{Y = y\})$$

can be written as

$$P(x|y)P(y) = P(x, y)$$

Marginal probability

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Example

Example

Measurements for the length and width of rectangular plastic covers for CDs are rounded to the nearest mm. Let X denote the length and Y denote the width. Assume that the joint probability of X and Y is represented by the following table

		x=length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

What is the probability distribution of X?

Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Example

Example

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Are X and Y independent?