MATH 205: Statistical methods

Lecture 19: Linear combinations of random variables

Multivariate distributions

Joint probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Marginal probability

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Example

Example

Measurements for the length and width of rectangular plastic covers for CDs are rounded to the nearest mm. Let X denote the length and Y denote the width. Assume that the joint probability of X and Y is represented by the following table

		x=length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

Are X and Y independent?

Expected value: multivariate

Given two discrete random variable X, Y and f is a function of (X, Y). Then $f(X, Y)$ is a also discrete random variable, and

$$\mathbb{E}[f(X,Y)] = \sum_{x,y} f(x,y)P(x,y)$$

Covariance

Definition

The covariance of of two random variables X and Y is

$$cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

Covariance: properties

- Note that
$$cov(X, X) = var(X)$$
- The covariance of of two random variables X and Y can be computed as
$$cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Example

Example

Assume that the joint probability of X (receive values 1, 2) and Y (receives values 1, 2, 3) is represented by the following table

X \ Y	1	2	3
1	0.14	0.42	0.06
2	0.06	0.28	0.04

Compute Cov(X, Y).

Independent variables have zero covariance

Proposition

If X and Y are independent, then

- $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- $cov(X, Y) = 0$

Variance of sum of independent variables

Proposition

If X and Y are independent, then

$$var(X+Y) = var(X) + var(Y)$$

Exercise

Problem

A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

Let Y be a continuous r.v. with density function

$$f(y) = \begin{cases} 2x & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Assume that X and Y are independent. Compute $E(X+Y)$, $Var(X+Y)$