MATH 205: Statistical methods

Chapter 5: Useful distributions

Joint probability and marginal probability

Definition 4.4 (Joint Probability Distribution of Two Discrete Random Variables) Assume we have two random variables X and Y. The probability that X takes the value x and Y takes the value y could be written as $P(\{X = x\} \cap \{Y = y\})$. It is more usual to write it as

$$P(x, y).$$

This is referred to as the **joint probability distribution** of the two random variables (or, quite commonly, the **joint**). You can think of this as a table of probabilities, one for each possible pair of x and y values.

Definition 4.6 (Marginal Probability of a Random Variable) Write $P(x, y)$ for the joint probability distribution of two random variables X and Y. Then

$$P(x) = \sum_y P(x, y) = \sum_y P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})$$

is referred to as the **marginal probability distribution** of X.

Independent variables

Definition 4.7 (Independent Random Variables) The random variables X and Y are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all values x and y. This means that

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\}),$$

which we can rewrite as

$$P(x, y) = P(x)P(y)$$

Expected value: multivariate

Given two discrete random variable X, Y and f is a function of (X, Y). Then $f(X, Y)$ is a also discrete random variable, and

$$\mathbb{E}[f(X, Y)] = \sum_{x,y} f(x, y)P(x, y)$$

Covariance

Definition

The covariance of of two random variables X and Y is

$$cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

- Note that
$$cov(X, X) = var(X)$$
- The covariance of of two random variables X and Y can be computed as
$$cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Typical example

Example

Assume that the joint probability of X (receive values 1, 2) and Y (receives values 1, 2, 3) is represented by the following table

X \ Y	1	2	3
1	0.14	0.42	0.06
2	0.06	0.28	0.04

Compute Cov(X, Y).

Variance of sum of independent variables

Proposition

If X and Y are independent, then

- $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- $cov(X, Y) = 0$

Proposition

If X and Y are independent, then

$$var(X + Y) = var(X) + var(Y)$$

Previous lecture: Exercise

Problem

A random variable X has the following pmf table

X	0	1	2
probability	0.25	0.5	0.25

Let Y be a continuous r.v. with density function

$$f(y) = \begin{cases} 2x & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Assume that X and Y are independent. Compute $E(X+Y)$, $Var(X+Y)$

Linear combination of random variables

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables (with possibly different means and/or variances). Define

$$T = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

then the mean and the standard deviation of T can be computed by

- $E(T) = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$
- $Var(T) = a_1^2Var(X_1) + a_2^2Var(X_2) + \ldots + a_n^2Var(X_n)$

Example

Problem

A gas station sells three grades of gasoline: regular unleaded, extra unleaded, and super unleaded. These are priced at 2.20, 2.35, and 2.50 per gallon, respectively.
Let X_1, X_2, and X_3 denote the amounts of these grades purchased (gallons) on a particular day. Suppose the X_i's are independent with $\mu_1 = 1000$, $\mu_2 = 500$, $\mu_3 = 300$, $\sigma_1 = 100$, $\sigma_2 = 80$, $\sigma_3 = 50$. Compute the expected value and the standard deviation of the revenue from sales

$$Y = 2.2X_1 + 2.35X_2 + 2.5X_3.$$

Example

Problem

Let X_1, X_2, X_3 *be three continuous r.v. sampled from the same distribution with density function*

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Let

$$Y = \frac{X_1 + X_2 + X_3}{3}$$

Compute $E[Y]$ *and* $Var[Y]$.

Review: Chapter 4 – Random variables and expectations

4.1 Random variables and probability distribution

- Discrete
- Continuous
- Joint and marginal distributions
- Independent variables

4.2 Expectations

- Mean
- Variance
- Covariance
- Expectations of functions of random variable(s)

Chapter 5: Useful distributions

Useful distributions

In the lectures:

- Uniform distribution
- Normal distribution
- Bernoulli distribution
- Binomial distribution

In the lab:

- Geometric distribution
- Poisson distribution
- Beta distribution
- Gamma distribution
- Exponential distribution

The Discrete Uniform Distribution

Definition

A random variable has the discrete uniform distribution if it takes each of k values with the same probability $1/k$, and all other values with probability zero.

Problem

Consider a random variable X that follows discrete uniform distribution on the set $\{1, 2, 3, 4\}$.
Compute $E(X)$ and $Var(X)$.