MATH 205: Statistical methods

Lecture 25: One-sided confidence intervals

Assumption: Normal distribution with known σ

- Using

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ is correct 95% of the time.

- If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

Confidence interval: principles

- We have

$$P\left[-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right] = 0.95$$

- Rearranging the inequalities gave

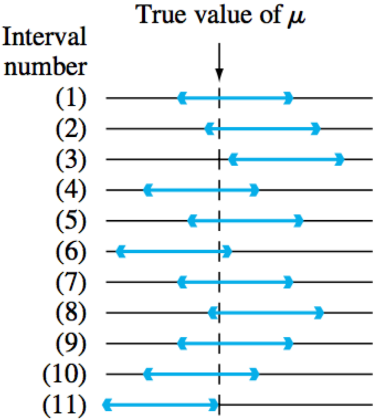
$$P\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

- This means that if you use

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

as a range to estimate μ, then you are correct 95% of the time.

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Interpreting confidence intervals

- Writing
$$P[\mu \in (\bar{X} - 1.7, \bar{X} + 1.7)] = 95\%$$
is okay.
- If $\bar{x} = 2.7$, writing
$$P[\mu \in (1, 4.4)] = 95\%$$
is NOT correct.
- Saying $\mu \in (1, 4.4)$ with confidence level 95% is good.
- Saying "if we repeat the experiment many times, the interval contains μ about 95% of the time" is perfect.

z-critical value

NOTATION

z_α will denote the value on the measurement axis for which α of the area under the z curve lies to the right of z_α. (See Figure 4.19.)

For example, $z_{.10}$ captures upper-tail area .10 and $z_{.01}$ captures upper-tail area .01.

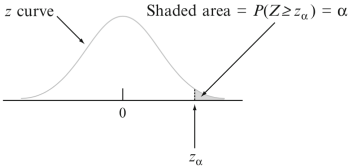


Figure 4.19 z_α notation illustrated

Since α of the area under the standard normal curve lies to the right of z_α, $1 - \alpha$ of the area lies to the left of z_α. Thus z_α *is the* $100(1-\alpha)$*th percentile of the standard normal distribution*. By symmetry the area under the standard normal curve to the left of $-z_\alpha$ is also α. The z_α's are usually referred to as ***z* critical values**. Table 4.1 lists the most useful standard normal percentiles and z_α values.

$100(1-\alpha)\%$ confidence interval

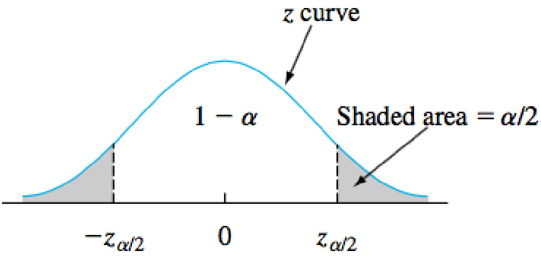


Figure 8.4 $P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha$

$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Example 1

Example

Assume that the helium porosity (in percentage) of coal samples taken from any particular seam is normally distributed with true standard deviation $\sigma = .75$.

- Compute a 95% CI for the true average porosity of a certain seam if the average porosity for 20 specimens from the seam was 4.85.
- How large a sample size is necessary if the width of the 95% interval is to be .40?

Example 1b

Example

Assume that the helium porosity (in percentage) of coal samples taken from any particular seam is normally distributed with true standard deviation $\sigma = .75$.

- Compute a 90% CI for the true average porosity of a certain seam if the average porosity for 20 specimens from the seam was 4.85.
- How large a sample size is necessary if the width of the 90% interval is to be .40?

One-sided CIs (Confidence bounds)

Example: One-sided confidence interval

Problem

Suppose the sediment density (g/cm) of a randomly selected specimen from a certain region is normally distributed with mean μ *(unknown) and standard deviation* $\sigma = 0.85$*. A random sample of* $n = 25$ *specimens is selected with sample average* $\bar{X}$*.*
Find a number b such that

$$P\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < b\right] = 0.95$$

CIs vs. one-sided CIs

CIs:

- $100(1-\alpha)\%$ confidence

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

One-sided CIs:

- $100(1-\alpha)\%$ confidence

$$\left(-\infty, \bar{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

- 95% confidence

$$\left(-\infty, \bar{x} + 1.64\frac{\sigma}{\sqrt{n}}\right)$$

$\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997

Example

Example

A sample of 66 obese adults was put on a low-carbohydrate diet for a year. The (sample) average weight loss was 7.7 lb. It is known that the standard deviation of weight loss is 19.0lb. Calculate a 99% lower confidence bound for the true average weight loss. From the result, do you think that the diet is effective?