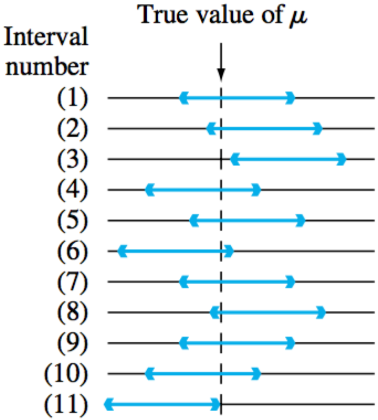
MATH 205: Statistical methods

Lecture 27: CIs of small-sample normal populations

Interpreting confidence intervals



95% confidence interval: If we repeat the experiment many times, the interval contains μ about 95% of the time

Confidence intervals for a population mean

- Normal distribution with known σ (Lecture 24)
  - Normal distribution
  - σ is known
- Large-sample confidence intervals (Lecture 25)
  - ~~Normal distribution~~

    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~

    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$
- Intervals based on normal distributions (this lecture)
  - Normal distribution
  - ~~σ is known~~

$\rightarrow$ Introducing t-distribution

Normal distribution with known σ

Assumption: Normal distribution with known σ

- Using

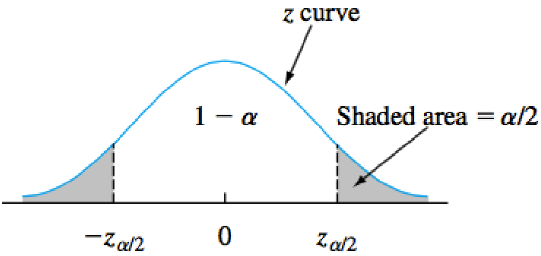
  $$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

  as a range to estimate μ is correct 95% of the time.
- If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$, we compute the observed sample mean $\bar{x}$. Then

  $$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

  is a 95% confidence interval of μ

$100(1-\alpha)\%$ confidence interval



$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha$$

$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

Large-sample CIs

Sample standard deviation

- Chapter 1: Given a data set $x_1, x_2, \ldots, x_n$, the sample standard deviation, denoted by s, is given by

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

- As a random variable:

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}}$$

Note: when n is small, it is recommended that we use

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}, \quad s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \ldots$, $X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

Normal distributions with small sample size

Assumptions

- the population of interest is normal
  (i.e., $X_1, \ldots, X_n$ constitutes a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$).
- σ is unknown
- When $n < 40$, S is no longer close to σ. Thus

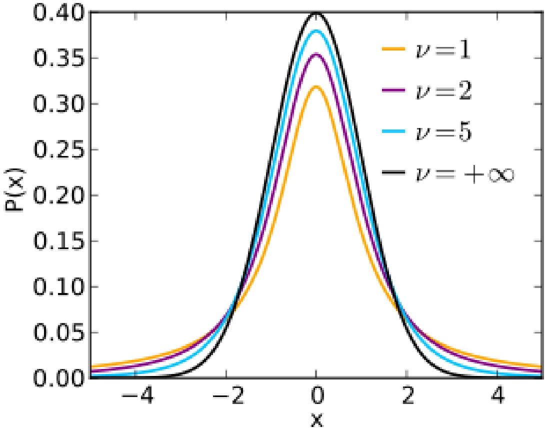
  $$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

  does not follow the standard normal distribution.

t distributions with degree of freedom ν

Probability density function

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$



t distributions

PROPERTIES OF T DISTRIBUTIONS

1. Each t_ν curve is bell-shaped and centered at 0.
2. Each t_ν curve is more spread out than the standard normal (z) curve.
3. As ν increases, the spread of the t_ν curve decreases.
4. As $\nu \to \infty$, the sequence of t_ν curves approaches the standard normal curve (so the z curve is often called the t curve with df $= \infty$).

t distributions

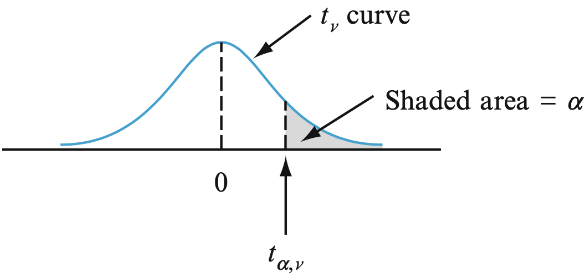
When $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ, the rv

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has the t distribution with $n - 1$ degree of freedom (df).

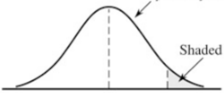
t distributions

Let $t_{\alpha,\nu}$ = the number on the measurement axis for which the area under the t curve with ν df to the right of $t_{\alpha,\nu}$, is α; $t_{\alpha,\nu}$ is called a **t critical value.**



$\alpha \rightarrow t$

Table A.5 Critical Values for t Distributions



ν \ α	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965

Confidence intervals

Let $\bar{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal population with mean μ. Then a **100(1 − α)% confidence interval for μ**, the **one-sample t CI**, is

$$\left(\bar{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}\right) \tag{8.15}$$

or, more compactly, $\bar{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$.

An **upper confidence bound for μ** is

$$\bar{x} + t_{\alpha,n-1} \cdot \frac{s}{\sqrt{n}}$$

and replacing + by − in this latter expression gives a **lower confidence bound for μ**; both have confidence level 100(1 − α)%.

Example 1

Example

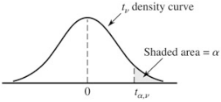
Let X be the amount of butterfat in pounds produced by a typical cow during a 305-day milk production period between her first and second calves. Assume that the distribution of X is $N(\mu, \sigma^2)$. To estimate μ, a farmer measured the butterfat production for n $=$ 20 cows and obtained the following data:

481	537	513	583	453	510	570	500	457	555
618	327	350	643	499	421	505	637	599	392

- Construct a 90% confidence interval for μ.
- Find a 90% one-sided confidence interval that provides an upper bound for μ.

$\alpha \to t$

Table A.5 Critical Values for *t* Distributions



	α						
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17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883

Example 2

Example

Here is a sample of ACT scores for students taking college freshman calculus:

24.00	28.00	27.75	27.00	24.25	23.50	26.25
24.00	25.00	30.00	23.25	26.25	21.50	26.00
28.00	24.50	22.50	28.25	21.25	19.75	

Assume that ACT scores are normally distributed, calculate a two-sided 95% confidence interval for the population mean.