MATH 205: Statistical methods

Lecture 28: Confidence intervals – notes

Confidence intervals for a population mean

- Normal distribution with known σ (Lecture 24–25)
  - Normal distribution
  - σ is known
- Large-sample confidence intervals (Lecture 26)
  - ~~Normal distribution~~
    $\rightarrow$ use Central Limit Theorem $\rightarrow$ needs $n > 30$
  - ~~σ is known~~
    $\rightarrow$ replace σ by s $\rightarrow$ needs $n > 40$
- Intervals based on normal distributions (this lecture)
  - Normal distribution
  - ~~σ is known~~

$\rightarrow$ Introducing t-distribution

$100(1-\alpha)\%$ confidence interval

A **$100(1-\alpha)\%$ confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \tag{8.5}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$.

$100(1-\alpha)\%$ confidence interval

If after observing $X_1 = x_1$, $X_2 = x_2, \dots$, $X_n = x_n$ $(n > 40)$, we compute the observed sample mean $\bar{x}$ and sample standard deviation s. Then

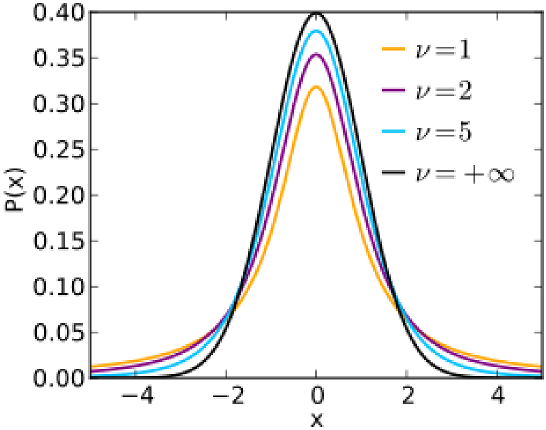
$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

is a 95% confidence interval of μ

t distributions with degree of freedom ν

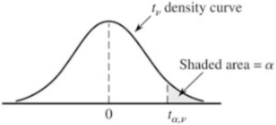
Probability density function

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$



$\alpha \rightarrow t$

Table A.5 Critical Values for t Distributions



ν \ α	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965

Confidence intervals

Let $\bar{x}$ and s be the sample mean and sample standard deviation computed from the results of a random sample from a normal population with mean μ. Then a **100(1 − α)% confidence interval for μ**, the **one-sample t CI**, is

$$\left(\bar{x} - t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}\right) \tag{8.15}$$

or, more compactly, $\bar{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$.

An **upper confidence bound for μ** is

$$\bar{x} + t_{\alpha,n-1} \cdot \frac{s}{\sqrt{n}}$$

and replacing + by − in this latter expression gives a **lower confidence bound for μ**; both have confidence level $100(1 - \alpha)\%$.

Example 1

Example

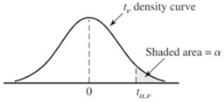
Let X be the amount of butterfat in pounds produced by a typical cow during a 305-day milk production period between her first and second calves. Assume that the distribution of X is $N(\mu, \sigma^2)$. To estimate μ, a farmer measured the butterfat production for n $= 20$ cows and obtained the following data:

481	537	513	583	453	510	570	500	457	555
618	327	350	643	499	421	505	637	599	392

- Construct a 90% confidence interval for μ.
- Find a 90% one-sided confidence interval that provides an upper bound for μ.

$\alpha \rightarrow t$

Table A.5 Critical Values for t Distributions



	α						
ν	.10	.05	.025	.01	.005	.001	.0005
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17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883

Example 2

Example

Here is a sample of ACT scores for students taking college freshman calculus:

24.00	28.00	27.75	27.00	24.25	23.50	26.25
24.00	25.00	30.00	23.25	26.25	21.50	26.00
28.00	24.50	22.50	28.25	21.25	19.75	

Assume that ACT scores are normally distributed, calculate a two-sided 95% confidence interval for the population mean.