MATH 205: Statistical methods

Lecture 32: Comparing the mean of two populations

Chapter 7: Significance of evidence

Last week: Test about a population mean

- Null hypothesis
$$H_0 : \mu = \mu_0$$
- The alternative hypothesis will be either:
  - $H_a : \mu > \mu_0$
  - $H_a : \mu < \mu_0$
  - $H_a : \mu \neq \mu_0$

Note: μ_0 here denotes a constant, and μ denotes the population mean (unknown)

Test about a population mean

- Test statistic

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

  or

$$\frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

- If under the assumption of the experiment, the test statistics above follow normal distribution, we will perform a z-test
- If under the assumption of the experiment, the test statistics above follow t-distribution, we will perform a t-test (with degree of freedom $n - 1$)

P-values for z-tests

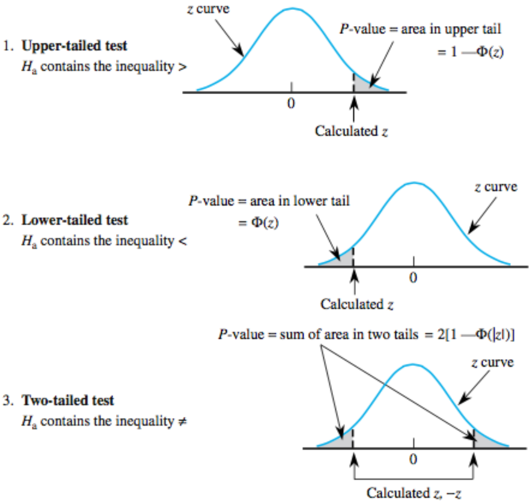

Figure 9.7 Determination of the *P*-value for a *z* test

Two-sample inference: example

Example

Let μ_1 and μ_2 denote true average decrease in cholesterol for two drugs. From two independent samples $X_1, X_2, \ldots, X_m$ and Y_1, $Y_2, \ldots, Y_n$, we want to test:

$$H_0 : \mu_1 = \mu_2$$
$$H_a : \mu_1 \neq \mu_2$$

Two-sample inference: example

Example

Let μ_1 and μ_2 denote true average decrease in cholesterol for two drugs. From two independent samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$, we want to test:

$$H_0 : \mu_1 - \mu_2 = 15$$
$$H_a : \mu_1 - \mu_2 > 15$$

Settings

Assumption

1. $X_1, X_2, \ldots, X_m$ *is a random sample from a population with mean* μ_1 *and variance* σ_1^2.
2. $Y_1, Y_2, \ldots, Y_n$ *is a random sample from a population with mean* μ_2 *and variance* σ_2^2.
3. *The* X *and* Y *samples are independent of each other.*

Analysis

Problem

Assume that

- $X_1, X_2, \ldots, X_m$ *is a random sample from a population with mean* μ_1 *and variance* σ_1^2.
- $Y_1, Y_2, \ldots, Y_n$ *is a random sample from a population with mean* μ_2 *and variance* σ_2^2.
- *The* X *and* Y *samples are independent of each other.*

Compute (in terms of $\mu_1, \mu_2, \sigma_1, \sigma_2, m, n$*)*

(a) $E[\bar{X} - \bar{Y}]$

(b) $Var[\bar{X} - \bar{Y}]$ *and* $\sigma_{\bar{X}-\bar{Y}}$

Properties of $\bar{X} - \bar{Y}$

Proposition

The expected value of $\overline{X} - \overline{Y}$ is $\mu_1 - \mu_2$, so $\overline{X} - \overline{Y}$ is an unbiased estimator of $\mu_1 - \mu_2$. The standard deviation of $\overline{X} - \overline{Y}$ is

$$\sigma_{\overline{X}-\overline{Y}} = \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

Confidence intervals

Assume further that the distributions of X and Y are normal and σ_1, σ_2 are known:

Problem

(a) *What is the distribution of*

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$$

(b) *Compute*

$$P\left[-1.96 \leq \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \leq 1.96\right]$$

(c) *Construct a 95% CI for $\mu_1 - \mu_2$ (in terms of $\bar{x}$, $\bar{y}$, m, n, σ_1, σ_2).*

Confidence intervals

When both population distributions are normal, standardizing $\overline{X} - \overline{Y}$ gives a random variable Z with a standard normal distribution. Since the area under the z curve between $-z_{\alpha/2}$ and $z_{\alpha/2}$ is $1 - \alpha$, it follows that

$$P\left(-z_{\alpha/2} < \frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} < z_{\alpha/2}\right) = 1 - \alpha$$

Manipulation of the inequalities inside the parentheses to isolate $\mu_1 - \mu_2$ yields the equivalent probability statement

$$P\left(\overline{X} - \overline{Y} - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}} < \mu_1 - \mu_2 < \overline{X} - \overline{Y} + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right) = 1 - \alpha$$

Testing the difference between two population means

- Setting: independent normal random samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ with known values of σ_1 and σ_2. Constant Δ_0.
- Null hypothesis:

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

- Alternative hypothesis:
  (a) $H_a : \mu_1 - \mu_2 > \Delta_0$
  (b) $H_a : \mu_1 - \mu_2 < \Delta_0$
  (c) $H_a : \mu_1 - \mu_2 \neq \Delta_0$
- When $\Delta = 0$, the test (c) becomes

$$H_0 : \mu_1 = \mu_2$$
$$H_a : \mu_1 \neq \mu_2$$

Testing the difference between two population means

Assume that we want to test the null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$ against each of the following alternative hypothesis

(a) $H_a : \mu_1 - \mu_2 > \Delta_0$

(b) $H_a : \mu_1 - \mu_2 < \Delta_0$

(c) $H_a : \mu_1 - \mu_2 \neq \Delta_0$

We use the test statistic:

$$z = \frac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}.$$

and derive the p-value in the same way as the one-sample tests.

Practice problem

Each student in a class of 21 responded to a questionnaire that requested their GPA and the number of hours each week that they studied. For those who studied less than 10 h/week the GPAs were

$$2.80, 3.40, 4.00, 3.60, 2.00, 3.00, 3.47, 2.80, 2.60, 2.00$$

and for those who studied at least 10 h/week the GPAs were

$$3.00, 3.00, 2.20, 2.40, 4.00, 2.96, 3.41, 3.27, 3.80, 3.10, 2.50$$

Assume that the distribution of GPA for each group is normal and both distributions have standard deviation $\sigma_1 = \sigma_2 = 0.6$. Treating the two samples as random, is there evidence that true average GPA differs for the two study times? Carry out a test of significance at level .05.