MATH 450 **Name (Print):** ____________________

Practice problems

Time Limit:

This exam contains 4 pages (including this cover page) and 3 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are allowed to bring a one-sided A4-sized hand-written note as reference.
You may use calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	0	
2	0	
3	0	
Total:	0	

Do not write in the table to the right.

1. For each part, write a brief explanation in the space provided. For true/false questions, also circle true/false.

   (a) **True or false**? For an unbiased estimator $\hat{\theta}$, the mean squared error of $\hat{\theta}$ is just the variance of the estimator $\hat{\theta}$.

   True. We have:

   $$MSE(\hat{\theta}) = Var(\hat{\theta}) + (bias)^2$$

   (the variance-bias decomposition). For an unbiased estimator, we know that $bias = 0$, which implies $MSE(\hat{\theta}) = Var(\hat{\theta})$.

   (b) Consider the distribution P with the following probability mass function

x	30	35	40
p(x)	0.2	0.3	0.5

   Let X_1, X_2 be a random sample of size 2 from P, and $T = X_1 - X_2$.
   Compute the expected value and the standard deviation of T.
   We first compute the expected value and variance of X:

   $$E[X] = 36.5, \quad Var(X) = 15.25$$

   Thus

   $$E[T] = E[X_1] - E[X_2] = 0,$$
   $$Var[T] = Var[X_1] + Var[X_2] = 30.5$$

   and

   $$\sigma_T = \sqrt{30.5}$$

2. Let $X_1, X_2, \ldots, X_n$ represent a random sample from a distribution with pdf

$$f(x, \theta) = \frac{2x}{\theta + 1} e^{-x^2/(\theta+1)}, \quad x > 0$$

(a) It can be shown that

$$E(X^2 - 1) = \theta$$

Use this fact to construct an estimator of θ based on the method of moments. We note that

$$E[X^2 - 1] = E[X^2] - 1,$$

thus

$$E[X^2] = \theta + 1$$

Using the method of moments

$$\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{n} = E[X^2] = \theta + 1$$

The estimator is

$$\hat{\theta} = \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{n} - 1.$$

(b) Derive the maximum-likelihood estimator for parameter θ based on the following dataset with $n = 10$

17.85, 11.23, 14.43, 19.27, 5.59, 6.36, 9.41, 6.31, 13.78, 11.95

The joint density function is

$$L(\theta) = f(x_1, x_2, \ldots, x_n, \theta) = x_1 x_2 \ldots x_n \left(\frac{2}{\theta + 1}\right)^n e^{-\frac{x_1^2 + x_2^2 + \ldots x_n^2}{\theta+1}}$$

Transform to log-scale

$$\ell(\theta) = \ln(L(\theta)) = \ln(x_1 x_2 \ldots x_n) + n \ln\left(\frac{2}{\theta + 1}\right) - \frac{x_1^2 + x_2^2 + \ldots x_n^2}{\theta + 1}$$

Take derivative with respect to θ:

$$\ell'(\theta) = -\frac{n}{\theta + 1} + \frac{x_1^2 + x_2^2 + \ldots x_n^2}{(\theta + 1)^2}$$

Set $\ell'(\theta) = 0$ to obtain

$$\hat{\theta} = \frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n} - 1.$$

Plug the numbers about in to obtain the point estimate.

3. Let X equal the amount of butterfat in pounds produced by a typical cow during a 305-day milk production period between her first and second calves. Assume that the distribution of X is $\mathcal{N}(\mu, \sigma^2)$, where $\sigma = 80$. To estimate μ, a farmer measured the butterfat production for n = 20 cows and obtained the following data:

481	537	513	583	453	510	570	500	457	555
618	327	350	643	499	421	505	637	599	392

Construct a 90% confidence interval for μ.

The confidence interval is:

$$\left(\bar{x} - z_{0.05}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{0.05}\frac{\sigma}{\sqrt{n}}\right)$$

In our case, $n = 20$, $\sigma = 80$, $z_{0.05} = 1.645$