Mathematical statistics

March 11^{th}, 2018

Lecture 12: Method of moments

Where are we?

Week 1	Probability reviews
Week 2	Chapter 6: Statistics and Sampling Distributions
Week 4	**Chapter 7: Point Estimation**
Week 7	Chapter 8: Confidence Intervals
Week 10	Chapter 9: Test of Hypothesis
Week 14	Regression

Overview

7.1 Point estimate
- unbiased estimator
- mean squared error

7.2 Methods of point estimation
- method of moments
- method of maximum likelihood.

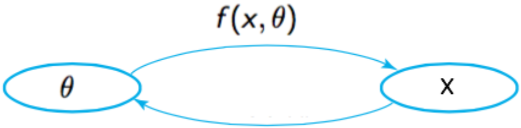
7.3 Sufficient statistic

7.4 Information and Efficiency
- Large sample properties of the maximum likelihood estimator

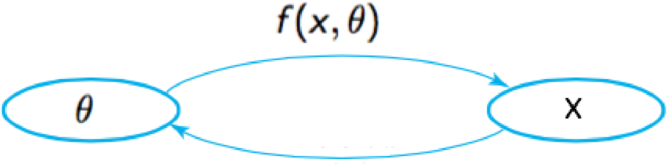
- Bootstrap

Question of this chapter

- Given a random sample $X_1, \ldots, X_n$ from a distribution with pmf/pdf $f(x, \theta)$ parameterized by a parameter θ
- Goal: Estimate θ



Point estimate



Definition

A point estimate $\hat{\theta}$ of a parameter θ is a single number that can be regarded as a sensible value for θ.

$$\begin{array}{ccccc} \text{population parameter} & \Longrightarrow & \textit{sample} & \Longrightarrow & \textit{estimate} \\ \theta & \Longrightarrow & X_1, X_2, \ldots, X_n & \Longrightarrow & \hat{\theta} \end{array}$$

Mean Squared Error

- Measuring error of estimation

$$|\hat{\theta} - \theta| \quad \text{or} \quad (\hat{\theta} - \theta)^2$$

- The error of estimation is random

Definition

The mean squared error of an estimator $\hat{\theta}$ is

$$E[(\hat{\theta} - \theta)^2]$$

Bias-variance decomposition

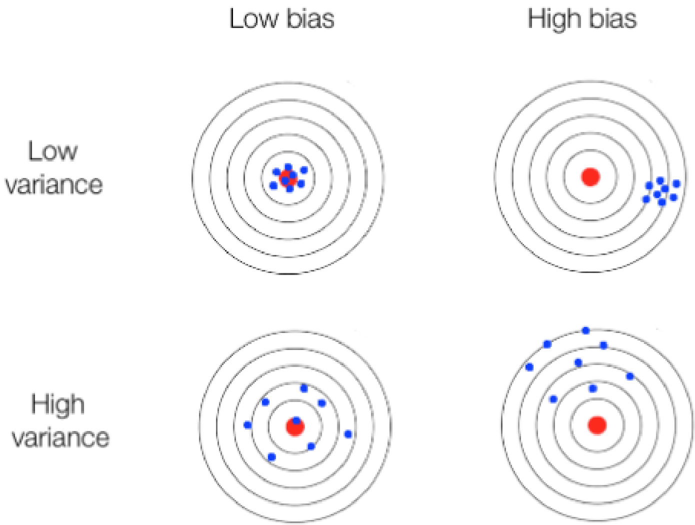
Theorem

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + \left(E(\hat{\theta}) - \theta\right)^2$$

Bias-variance decomposition

Mean squared error = variance of estimator + $(bias)^2$

Bias-variance decomposition



Unbiased estimators

Definition

A point estimator $\hat{\theta}$ is said to be an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

for every possible value of θ.

Unbiased estimator

$\Leftrightarrow$ Bias $= 0$

$\Leftrightarrow$ Mean squared error $=$ variance of estimator

Method of moments

Example

Problem

Let $X_1, \ldots, X_{15000}$ be a random sample from the exponential distribution with parameter λ, that is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

Provide an estimator of λ.

Exponential distribution

Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
PDF	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Quantile	$-\ln(1 - F) / \lambda$
Mean	λ^{-1} $(= \beta)$
Median	$\lambda^{-1}\ln(2)$
Mode	0
Variance	λ^{-2} $(= \beta^2)$
Skewness	2
Ex. kurtosis	6
Entropy	$1 - \ln(\lambda)$
MGF	$\frac{\lambda}{\lambda - t}, \text{ for } t < \lambda$
CF	$\frac{\lambda}{\lambda - it}$
Fisher information	λ^{-2}

Idea

- The expectation of a exponential random variable is

$$E[X] = \frac{1}{\lambda}$$

- For large n, we have

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

  is close to $E[X]$

- We can compute $\bar{x}$ from the data $\rightarrow$ approximate λ

Moments

- Let $X_1, \dots, X_n$ be a random sample from a normal distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \dots,$ the k^{th} population moment, or k^{th} moment of the distribution $f(x)$, is

$$E(X^k)$$

  - First moment: the mean
  - Second moment: $E(X^2)$

Sample moments

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf $f(x)$.
- For $k = 1, 2, 3, \ldots,$ the k^{th} sample moment is

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n}$$

The law of large numbers provides that when $n \to \infty$

$$\frac{X_1^k + X_2^k + \ldots + X_n^k}{n} \to E(X^k)$$

Method of moments: ideas

- Let $X_1, \ldots, X_n$ be a random sample from a distribution with pmf or pdf
$$f(x; \theta_1, \theta_2, \ldots, \theta_m)$$
- Assume that for $k = 1, \ldots, m$
$$\hat{u}_k = \frac{X_1^k + X_2^k + \ldots + X_n^k}{n} = E(X^k)$$
- Solve the system of equations for $\theta_1, \theta_2, \ldots, \theta_m$

Method of moments: Example 1

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from the exponential distribution with parameter λ, that is

$$f(x; \lambda) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

The observed data are

$$3.11, 0.64, 2.55, 2.20, 5.44,$$

$$3.42, 1.39, 8.13, 1.82, 1.30$$

Use the method of moments to obtain an estimator of λ.

- Equation: $k = 1$

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n} = E(X) = \frac{1}{\lambda}$$

- Solve the system of equations for λ

$$\lambda = \frac{1}{\bar{X}}$$

Method of moments: Example 2

Problem

Suppose that for a parameter $0 \leq \theta \leq 1$, X is the outcome of the roll of a four-sided tetrahedral die

x	1	2	3	4
$p(x)$	$\frac{3\theta}{4}$	$\frac{\theta}{4}$	$\frac{3(1-\theta)}{4}$	$\frac{(1-\theta)}{4}$

Suppose the die is rolled 10 times with outcomes

$$4, 1, 2, 3, 1, 2, 3, 4, 2, 3$$

Use the method of moments to obtain an estimator of θ.

Method of moments: Example 3

Problem

Let $X_1, \ldots, X_{10}$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} (\theta+1)x^{\theta} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

A random sample of ten students yields data

$$x_1 = .92,\ x_2 = .79,\ x_3 = .90,\ x_4 = .65,\ x_5 = .86,$$

$$x_6 = .47,\ x_7 = .73,\ x_8 = .97,\ x_9 = .94,\ x_{10} = .77$$

Use the method of moments to obtain an estimator of θ.

Method of moments: Example 4

Problem

Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $\mathcal{N}(0, \sigma^2)$.
Use the method of moments to obtain an estimator of σ.

Method of moments: Example 5

Problem

Let $\beta > 1$ and $X_1, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \begin{cases} \frac{\beta}{x^{\beta+1}} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the method of moments to obtain an estimator of β.

Minimum variance unbiased estimator (MVUE)

Definition

Among all estimators of θ that are unbiased, choose the one that has minimum variance. The resulting $\hat{\theta}$ is called the minimum variance unbiased estimator (MVUE) of θ.

Recall:

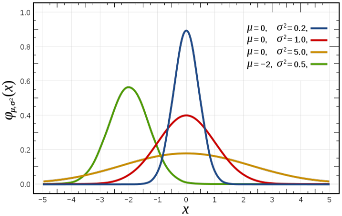
- Mean squared error $=$ variance of estimator $+$ $(bias)^2$
- unbiased estimator $\Rightarrow$ bias $=0$

$\Rightarrow$ MVUE has minimum mean squared error among unbiased estimators

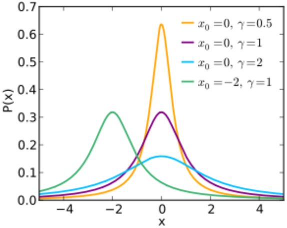
What is the best estimator of the mean?

Question: Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ. What is the best estimator of μ?

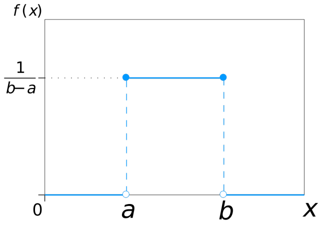
Example 7.8



$\mu=0, \quad \sigma^2=0.2,$
$\mu=0, \quad \sigma^2=1.0,$
$\mu=0, \quad \sigma^2=5.0,$
$\mu=-2, \quad \sigma^2=0.5,$
$\varphi_{\mu,\sigma^2}(x)$
x

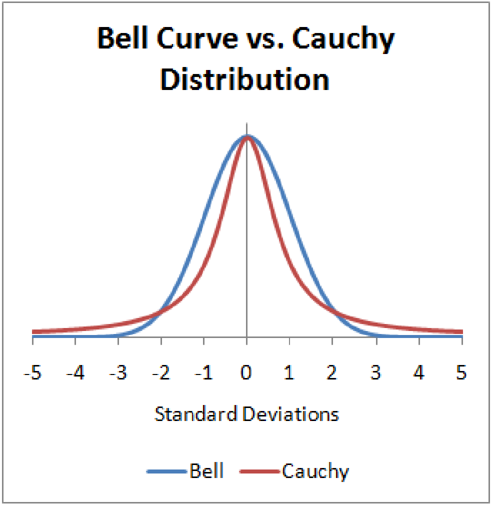


$x_0=0, \ \gamma=0.5$
$x_0=0, \ \gamma=1$
$x_0=0, \ \gamma=2$
$x_0=-2, \ \gamma=1$
P(x)
x



$f(x)$
$\frac{1}{b-a}$
0
a
b
x

Normal vs. Cauchy



What is the best estimator of the mean?

Question: Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ. What is the best estimator of μ?

Answer: It depends.

- Normal distribution $\rightarrow$ sample mean $\bar{X}$
- Cauchy distribution $\rightarrow$ sample median $\tilde{X}$
- Uniform distribution $\rightarrow$ no tails, uniform

$$\hat{X}_e = \frac{\text{largest number} + \text{smaller number}}{2}$$

- In all cases, 10% trimmed mean performs pretty well

MVUE of normal distributions

Theorem

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ. Then the estimator $\hat{\mu} = \bar{X}$ is the MVUE for μ.